

Macroeconomic Bond Risks at the Zero Lower Bound

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Abstract

Close-to-zero interest rates challenge standard economic models in which zero lower bound (ZLB) is absent. We estimate a recursive utility model which features time-varying latent expected real growth, expected inflation, and stochastic inflation volatility. Using an approximate solution to bond prices, we show that the ZLB model successfully captures interest rates in a ZLB period, without a deterioration in fit to rates in normal times. Incorporating ZLB lowers the estimates of expected inflation and increases inflation volatility. It leads to large, negative and volatile shadow rates, large and volatile shadow risk premia, and small and volatile lift-off probabilities.

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1 Introduction

From 2008 onwards, nominal interest rates have been at historical lows. Low levels of bond yields which persist for a prolonged period of time makes this period special and challenging for standard asset-pricing models. Intuitively, the rates cannot go below a zero lower bound (ZLB) as long as there is cash, i.e., an asset that provides a deterministic nominal return equal to zero.¹ However, most of the existing term-structure models do not entertain a zero lower bound (ZLB) restriction. These models are usually affine in the risk factors, and thus do not capture potential changes in bond price dynamics close to a zero bound. In this paper, we consider an economic model for the term structure which incorporates a lower bound restriction for the interest rates. We show that this restriction has significant implications for the estimates of latent macroeconomic risk factors and the implied dynamics of bond yields in the periods where the ZLB restriction is binding.

Specifically, we consider a long-run risks model for the nominal term structure, in the spirit of Bansal and Yaron (2004) and Bansal and Shaliastovich (2013). The economic environment features recursive utility and persistent fluctuations in expected consumption growth, expected inflation, and the volatility of expected inflation. As in Bansal and Shaliastovich (2013) and Piazzesi and Schneider (2006), expected inflation shocks are non-neutral and negatively affect expected consumption: high expected inflation is bad news for future real growth. The volatility of expected inflation shocks varies over time, which captures the fluctuations in the uncertainty about future inflation. With a preference for early resolution of uncertainty, the shocks in the three economic states are priced and affect equilibrium asset prices and risk premia. Expected consumption risks have a positive market price of risk, while shocks to expected inflation and volatility of inflation carry negative premia. That is, bad states of the economy are those with low expected consumption and high and volatile expected inflation.

Next we consider model implications for nominal bond prices. We first start with a benchmark ‘No ZLB’ model which does not entertain a ZLB. As shown in Bansal and Shaliastovich (2013), the main equilibrium quantities, like the risk-free rate and the wealth-consumption ratio are affine in the economic states. To solve the model in the presence of a ZLB restriction, we follow Krippner (2012) who suggests an approximate solution for the ZLB bond prices. In this approach, as in Black (1995), the short rate is equal to the

¹Strictly speaking, the lower bound for interest rates does not have to be zero, as one can currently observe in the Eurozone. The fact that rates can also take on negative values can, e.g., represent opportunity costs to holding cash.

maximum of the shadow interest rate and zero, and is guaranteed to be positive. Longer-term ZLB bond prices are constructed from the instantaneous forward rates over the life of the bond which are also forced to be positive. Unlike the exact solution, which would in a setting like ours require computationally demanding Monte Carlo simulations, this approximation is convenient and tractable. It involves the valuation of a call option on a bond, which can be efficiently implemented with multiple states, stochastic volatility, and non-Gaussian shocks, as in our setup. Notably, due to the presence of this call option component, bond prices and bond risk premia are no longer linear in the state variables.

We estimate the model in the data and quantitatively assess the importance of the ZLB restriction. For the estimation we use macroeconomic data on monthly real consumption growth, inflation, and the asset-price data on U.S. Treasury bills and bonds with maturities from three months to ten years over the period from 1987 to 2014. Using Bayesian particle Markov chain Monte Carlo (MCMC) methods, we first estimate a linear No ZLB model in a pre-2008 sample, and obtain a set of model parameters, filtered economic states, and implied model yields which correspond to the No ZLB model output. In the next step, we fix the parameters of the model at their estimated values, and consider the ‘ZLB’ model implications.² While we do not re-estimate the parameters of the model, we use a particle filter to extract a new set of economic states which correspond to the non-linear ZLB model. This allows us to assess the importance of the ZLB restriction for the model-implied yields and the estimates of the economic states.

Introducing the ZLB into the model has significant effects on the levels and dynamics of the yields and the state variables. Indeed, we document that while the economic states are essentially identical in the period when rates are away from the ZLB in the data, they can be quite different when the restriction is binding in what we call the ZLB period. Specifically, expected inflation drops, while inflation uncertainty increases in the ZLB relative to No ZLB model. Economically, the introduction of the ZLB restriction has two opposing effects on bond prices. First of all, it directly increases the level of the term structure, since it introduces a floor for interest rates. Furthermore, with a usually upward sloping term structure, rates for shorter maturities are affected more than longer-term yields, which dampens the slope of the term structure and decreases bond risk premia. On the other hand, the estimates of a lower expected inflation and higher inflation volatility implied by the ZLB model lead to lower levels of yields, higher slopes of the term structure, and

²This approach is justified, since the same ‘true’ model has to hold in the two regimes which differ with respect to the tightness of the ZLB restriction.

higher bond risk premia. We find that for short rates, incorporating the ZLB restriction plays a far more important role than the difference in the state variables, so short rates increase. For long rates, the two channels nearly offset each other, so the long-term yields and long-term bond risk premia are quite similar in the No ZLB and ZLB models. Overall, the ZLB model successfully captures the prices of short and long bonds. The ZLB yields are non-negative and stable in the ZLB periods, as in the data, and the ZLB and the No ZLB yields are similar away from the ZLB periods and at longer maturities.

Finally, we consider model implications for the ‘shadow’ interest rates and risk premia, the lift-off probabilities, and yield forecasts, where the shadow rate is the rate that would prevail if the lower bound of zero were not binding. We find that the shadow rates and risk premia are quite large and volatile. In the period from 2008 to 2014, the 3-month shadow rate averages -0.9% and has a volatility of 1.6% , while the shadow bond risk premium is about a quarter larger than the bond risk premia obtained in specifications where the ZLB is obeyed.

We further compute conditional lift-off probabilities that determine the likelihood of interest rates becoming positive over certain predetermined horizons. During the 2010-2014 period, the ZLB model assigns almost a zero chance that the short, 3-month rates would become positive within one month, while the 5-year lift-off probability is generally above 70% for the whole sample. Interestingly, the lift-off probabilities increase substantially if we base the inference on the state variables inferred from a bond pricing model which does not feature the ZLB. The reason is that the estimated states imply higher current and future yields and thus also a higher lift-off probability. Finally, we consider model forecasts for future yields at the end of 2014. We find that the model implies a slow recovery for interest rates, and assigns a 5% chance that the rates reach their steady state values by 2020. Notably, the uncertainty about these forecasts is quite considerable.

Related Literature Our paper is linked to several different areas of the literature. The fundamental idea of a zero lower bound for interest rates goes back to Black (1995), who also introduces the notion of a shadow rate. McCallum (2000) provides a theoretical foundation for the existence of this bound in the context of a macroeconomic model. He also emphasizes that the lower bound is only exactly equal to zero if there is no opportunity cost in holding cash. So rates could actually go below zero without generating arbitrage opportunities, but only to very small negative values, so that the assumption of a zero bound seems economically reasonable.

The fact that rates have been at historically low levels over a major part of the last

decade has sparked the interest of the academic community in the topic of the zero lower bound. One strand of the literature is dealing with the macroeconomic implications of near-zero interest rates. For example, Gavin, Keen, Richter, and Throckmorton (2013) and Aruoba, Cuba-Borda, and Schorfheide (2016) introduce the zero lower bound into a New Keynesian model and then explore the implied dynamics of the economy. Swanson and Williams (2014) investigate the sensitivity of U.S. bond yields to macroeconomic news, and Moessner (2013) performs a similar exercise for Canada and the UK. Wright (2012) empirically investigates the impact of short-term rates being stuck at the ZLB on longer term rates.

A number of papers have investigated the consequences of introducing a zero lower bound in reduced-form interest rate models, which take the stochastic process of the short rate or the latent factors behind it as given. While one could basically easily ensure a non-negative short rate by employing the popular square-root process for the short rate introduced by Cox, Ingersoll, and Ross (1985), this process does not provide a good fit to the yields when the short rate is close to zero. Based on this fact, Kim and Singleton (2012) suggest a quadratic Gaussian model and empirically test it with Japanese data, which are well suited for that purpose due to the extended spell of very low rates. Recently, Monfort, Pegoraro, Rennez, and Roussellet (2014) proposed the autoregressive Gamma-Zero process, which guarantees positive short rates and exhibits characteristics of the short rate near zero which are better in line with the data than in the case of the square-root process. Ichue and Ueno (2006) take a slightly different approach by specifying the dynamics of the short rate and inflation and apply it to an analysis of the monetary policy in Japan.

Besides models which automatically guarantee non-negative short rates there are also approaches which take the shadow rate process as given and then explicitly impose the restriction that the observed rate must be non-negative, i.e., it must be equal to the maximum of the shadow rate and zero. Priebisch (2013) suggests a standard Gaussian reduced-form term structure model and then presents an efficient technique to approximate the risk-neutral expectation of the (exponential of the) integral over the future (non-negative) short rates, which must be computed to obtain bond prices. This approach is also used in the empirical study by Kim and Priebisch (2013). Wu and Xia (2016) discuss related approximation methods. These techniques are indeed applicable to models with Gaussian short rates, since in this case truncated expectations of the form $E_t [\max\{r_{t+\tau}, 0\}]$ are particularly easy to compute. When the distribution of the shadow rate is not Gaussian, the

integral has to be estimated by Monte Carlo simulation, which is time-consuming and not very well suited for empirical applications when many bond prices have to be computed in the course of a model estimation. Due to this problem we do not follow this approach, but instead use the method developed by Krippner (2012). Specifically, we represent the difference between the prices of a zero-coupon bond under the model obeying the zero lower bound and the prices of certain call options on bonds under the shadow rate model. This method is also used by Christensen and Rudebusch (2015a) in an empirical study of again Japanese data, and the authors further show that the Krippner (2012) method delivers prices which are very close to the ones generated by Monte Carlo simulation of the shadow rate path.

Finally, our economic term-structure model follows the Bansal and Yaron (2004) long-run risks specification, similar to Bansal and Shaliastovich (2013). Eraker (2006), Piazzesi and Schneider (2006), Hasseltoft (2012) and Doh (2013) study economic implications for the bond yields and bond risk premia in a similar setting with recursive utility and persistent economic shocks. Wachter (2006), and Bekaert, Engstrom, and Xing (2009) consider the bond pricing implications in the context of the habits model. Creal and Wu (2016) consider recursive preferences together with a habit for current consumption. Recent works in the no-arbitrage and statistical literature include the affine models of Dai and Singleton (2002) and Duffee (2002), the regime-switching models of Bansal and Zhou (2002), and the macro-finance specifications of the term structure by Ang and Piazzesi (2003), Ang, Dong, and Piazzesi (2005), Rudebusch and Wu (2008), and Bikbov and Chernov (2010). All these papers, however, do not consider the effects of the zero lower bound.

The rest of the paper is organized as follows. In the next section, we present the economic model. The solution to the model and the term structures of interest rates with and without the zero lower bound are determined in Section 3. Section 4 gives the details on the estimation procedure and the resulting parameters. We discuss the quantitative implications of the models in Section 5. Section 6 concludes.

2 Model Setup

We consider a long-run risks type model for the nominal term structure. Our model builds on and extends existing specifications, such as Eraker (2006), Piazzesi and Schneider (2006), Hasseltoft (2012), and Bansal and Shaliastovich (2013), to accommodate a zero lower bound for the nominal rates.

2.1 Preferences

The investor's preferences over future consumption are described by the Kreps-Porteus, Epstein-Zin-Weil recursive utility function (see Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989)):

$$U_t = \left[(1 - e^{-\delta}) C_t^{1-\frac{1}{\psi}} + e^{-\delta} \mathbf{E}_t \left[U_{t+1}^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}},$$

with a coefficient of relative risk aversion γ , intertemporal elasticity of substitution ψ , and a subjective rate of time preference δ . Recursive preferences allow to disentangle risk aversion and intertemporal elasticity of substitution, which are the inverse of each other for standard CRRA preferences, i.e., there $\gamma = 1/\psi$.

In equilibrium, the log return r_{t+1} on any asset satisfies the Euler equation

$$\mathbf{E}_t [e^{m_{t+1}+r_{t+1}}] = 1,$$

where m_{t+1} is the logarithm of the stochastic discount factor over the time interval from t to $t+1$. With recursive preferences, the (real) stochastic discount factor is given by,

$$m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{c,t+1}, \quad (2.1)$$

where $\beta = e^{-\delta}$, $\theta = \frac{1-\gamma}{1-\frac{1}{\psi}}$, Δc denotes the growth rate of the logarithm of aggregate consumption, and r_c is the logarithm of the return on the claim to aggregate consumption. The return r_c is unobserved, and is determined endogenously using the dynamics of consumption and the investor's preferences.

2.2 Economy Dynamics

We consider an endowment economy in which real consumption growth and the rate of inflation are specified exogenously.

The dynamics of consumption growth Δc and of the inflation rate π are

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + x_{c,t} + \sigma_c \varepsilon_{c,t+1} \\ \pi_{t+1} &= \mu_\pi + x_{\pi,t} + \sigma_\pi \varepsilon_{\pi,t+1}. \end{aligned}$$

The expected growth rate of consumption and the expected inflation rate depend on x_c and x_π , respectively. Their dynamics are given by

$$\begin{aligned}x_{c,t+1} &= \rho_{xc}x_{c,t} + \rho_{xc,x\pi}x_{\pi,t} + \sqrt{\nu_{xc,t}}\varepsilon_{xc,t+1}, \\x_{\pi,t+1} &= \rho_{x\pi}x_{\pi,t} + \sqrt{\nu_{x\pi,t}}\varepsilon_{x\pi,t+1}.\end{aligned}$$

The expected growth rates are persistent and have time-varying volatilities. We follow Bansal and Shaliastovich (2013) and Piazzesi and Schneider (2006) and assume that expected inflation has an impact on future expected consumption growth, represented by the parameter $\rho_{xc,x\pi}$. Inflation is thus not necessarily neutral and will have real effects for $\rho_{xc,x\pi} \neq 0$. Finally, all economic shocks ε_c , ε_π , ε_{xc} and $\varepsilon_{x\pi}$ to consumption, inflation, expected consumption growth and expected inflation are standard normal.

The variances ν_{xc} and $\nu_{x\pi}$ of the innovations in x_c and x_π are stochastic. Their dynamics are given by

$$\begin{aligned}\nu_{xc,t+1} &= \rho_{\nu xc}\nu_{xc,t} + \varepsilon_{\nu xc,t+1} \\ \nu_{x\pi,t+1} &= \rho_{\nu x\pi}\nu_{x\pi,t} + \varepsilon_{\nu x\pi,t+1}.\end{aligned}$$

The variance innovations $\varepsilon_{\nu xc}$ and $\varepsilon_{\nu x\pi}$ follow Gamma distributions with shape parameters $k_{\nu xc}$ and $k_{\nu x\pi}$ and scale parameters $\theta_{\nu xc}$ and $\theta_{\nu x\pi}$, respectively. The non-negativity of the innovations guarantees that the variance processes stay non-negative. We choose the slope and scale parameters as follows:

$$k_{\nu xi} = \frac{(1 - \rho_{\nu xi})^2 \bar{\nu}_{xi}^2}{\sigma_{\nu xi}^2}, \quad \theta_{\nu xi} = \frac{\sigma_{\nu xi}^2}{(1 - \rho_{\nu xi})\bar{\nu}_{xi}} \quad (i = c, \pi),$$

where the $\bar{\nu}_{xi}$ are the unconditional means of the variances, and the $\sigma_{\nu xi}$ are the volatilities of the innovations. All shocks are assumed to be mutually independent.

In the empirical implementation, we focus on the time-variation in the volatility of expected inflation which has a substantially larger contribution to bond yields and bond risk premia, relative to the volatility of expected consumption growth.³ For parsimony, we therefore set the latter equal to a constant, i.e., $\nu_{xc,t} \equiv \bar{\nu}_{xc}$.

³See Bansal and Shaliastovich (2013) and Song (2014). Song (2014) further shows that using equity price data helps to better identify movements in real volatility.

3 Asset Prices

3.1 Stochastic Discount Factor

The solution of a No-ZLB model is standard, and follows Bansal and Shaliastovich (2013).

The stochastic discount factor (2.1) depends on the return r_c on the consumption claim. We rely on the Campbell-Shiller approximation

$$r_{c,t+1} = \kappa_0 + \kappa_1 wc_{t+1} - wc_t + \Delta c_{t+1}, \quad (3.1)$$

where κ_0 and κ_1 are linearizing coefficients and wc is the logarithm of the wealth-consumption ratio. In equilibrium, the latter is given by an affine function of the state variables:

$$wc_t = A_0 + A_{xc}x_{c,t} + A_{x\pi}x_{\pi,t} + A_{\nu xc}\nu_{xc,t} + A_{\nu x\pi}\nu_{x\pi,t}. \quad (3.2)$$

The loadings on the state variables are given in Appendix A.1. Plugging the approximation for r_c and the wealth-consumption ratio into Equation (2.1) gives the real stochastic discount factor.

To price nominal assets, we use a nominal Euler equation,

$$\mathbb{E}_t \left[e^{m_{t+1}^{\$} + r_{t+1}^{\$}} \right] = 1,$$

which holds for any nominal return $r_{t+1}^{\$}$. The logarithm of the nominal stochastic discount factor is given by

$$\begin{aligned} m_{t+1}^{\$} &= m_{t+1} - \pi_{t+1} \\ &= m_0^{\$} + m_{xc}^{\$}x_{c,t} + m_{x\pi}^{\$}x_{\pi,t} + m_{\nu xc}^{\$}\nu_{xc,t} + m_{\nu x\pi}^{\$}\nu_{x\pi,t} \\ &\quad - \lambda_c^{\$}\sigma_c\varepsilon_{c,t+1} - \lambda_{\pi}^{\$}\sigma_{\pi}\varepsilon_{\pi,t+1} - \lambda_{xc}^{\$}\sqrt{\nu_{xc,t}}\varepsilon_{xc,t+1} - \lambda_{x\pi}^{\$}\sqrt{\nu_{x\pi,t}}\varepsilon_{x\pi,t+1} \\ &\quad - \lambda_{\nu xc}^{\$}\varepsilon_{\nu xc,t+1} - \lambda_{\nu x\pi}^{\$}\varepsilon_{\nu x\pi,t+1}. \end{aligned}$$

The constant $m_0^{\$}$ and the loadings on the state variables are given in Appendix A.1. The

market prices of risk are:

$$\begin{aligned}
\lambda_c^\$ &= \gamma & \lambda_\pi^\$ &= 1 \\
\lambda_{xc}^\$ &= \kappa_1(1 - \theta)A_{xc} & \lambda_{x\pi}^\$ &= \kappa_1(1 - \theta)A_{x\pi} \\
\lambda_{\nu xc}^\$ &= \kappa_1(1 - \theta)A_{\nu xc} & \lambda_{\nu x\pi}^\$ &= \kappa_1(1 - \theta)A_{\nu x\pi}.
\end{aligned}$$

As with a CRRA utility, the market price of short-run consumption risk is equal to the relative risk aversion γ , and the market price of inflation risk is equal to one. Notably, under the CRRA utility specification ($\theta = 1$), only the short-run risks are priced, and the market prices of expected consumption, expected inflation, and volatility risks are all zero. With recursive utility these innovations affect the endogenous wealth of the agent and thus also marginal utility.

In particular, positive innovations to expected consumption growth are good news for the investor, and imply a higher wealth-consumption ratio for $\psi > 1$ (together with $\gamma > 1$), since the substitution effect dominates the wealth effect. A_{xc} is thus positive, and the market price of risk $\lambda_{xc}^\$$ is positive, too. The sign of $\lambda_{x\pi}^\$$ depends on how expected inflation affects the future dynamics of consumption. If its impact on expected future consumption growth is negative ($\rho_{xc,x\pi} < 0$), a rise in expected inflation is bad news for the investor, and $\lambda_{x\pi}^\$$ is negative. An asset with a positive exposure to expected inflation thus earns a negative risk premium for this risk, since it hedges against a worsening of economic conditions. The market prices of risk for variance are negative, i.e. $\lambda_{\nu xc}^\$ < 0$ and $\lambda_{\nu x\pi}^\$ < 0$.

3.2 No ZLB Model Solution

We first consider the term structure of interest rates if there is no zero lower bound. In this case, the model is affine, and the bond prices are exponentially linear functions of the state variables:

$$P(t, t + \tau) = e^{B_0(\tau) + B_{xc}(\tau)x_{c,t} + B_{x\pi}(\tau)x_{\pi,t} + B_{\nu xc}(\tau)\nu_{xc,t} + B_{\nu x\pi}(\tau)\nu_{x\pi,t}}. \quad (3.3)$$

The functions B_0 , B_{xc} , $B_{x\pi}$, $B_{\nu xc}$, and $B_{\nu x\pi}$ are given in Appendix A.2. The spot rates (i.e., the yields on the zero coupon bonds) $y(t, t + \tau) = -\frac{1}{\tau} \ln P(t, t + \tau)$ are then given as

$$y(t, t + \tau) = -\frac{B_0(\tau) + B_{xc}(\tau)x_{c,t} + B_{x\pi}(\tau)x_{\pi,t} + B_{\nu xc}(\tau)\nu_{xc,t} + B_{\nu x\pi}(\tau)\nu_{x\pi,t}}{\tau}. \quad (3.4)$$

They are affine functions of the state variables. Since expected consumption growth rate x_c and expected inflation x_π are not restricted with respect to their signs, interest rates can be negative.

The risk premium on the bond depends on the sensitivities B of the bond and on the market prices of risk. For a bond with initial time to maturity $\tau + 1$, the risk premium is approximately equal to

$$\begin{aligned} & \mathbb{E}_t \left[\frac{P(t+1, t+\tau+1)}{P(t, t+\tau+1)} \right] - e^{r_t} \\ & \approx B_{x_c}(\tau) \lambda_{x_c}^\$ \nu_{x_c, t} + B_{x_\pi}(\tau) \lambda_{x_\pi}^\$ \nu_{x_\pi, t} + B_{\nu_{x_c}}(\tau) \lambda_{\nu_{x_c}}^\$ + B_{\nu_{x_\pi}}(\tau) \lambda_{\nu_{x_\pi}}^\$. \end{aligned}$$

It is an affine function of the variances of the state variables, i.e. of ν_{x_c} and ν_{x_π} .

3.3 ZLB Model Solution

With a zero lower bound on interest rates, the bond prices and yields in Equations (3.3) and (3.4) describe the shadow term structure of interest rates. Shadow interest rates can become negative and thus violate a zero lower bound for the observed nominal rates.

To determine the term structure in case of a zero lower bound, Black (1995) suggests to directly adjust the short-rate in a given interest rate model. The starting point of his approach is a model for the shadow short rate r_t (i.e., the interest rate from t to $t+1$) and a given risk-neutral measure \mathbb{Q} . The true short-rate at t is then set equal to $\max\{r_t, 0\}$, i.e. the shadow short-rate is replaced by the lower bound of zero whenever it violates this bound. Economically, this reflects that the investor will hold cash instead of bonds whenever interest rates are negative. The resulting prices of zero coupon bonds are

$$\underline{P}(t, T) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\sum_{s=t}^{T-1} \max\{r_s, 0\}} \right].$$

In general, they have to be solved numerically even if the shadow rate model is affine.

Krippner (2012) suggests an approximation to determine the ZLB yields which keeps the analytical tractability of affine models. He first determines the price of an auxiliary zero-coupon bond for which the non-negativity condition on the short rate is imposed over the last period. The price of this auxiliary bond with maturity in $T+1$ is

$$P_a(t, T+1) = \mathbb{E}_t \left[M_{t,T}^\$ \min\{1, P(T, T+1)\} \right]. \quad (3.5)$$

The forward rate at time t for the interval $[T, T + 1]$ is then given by

$$\underline{f}(t, T) = -[\ln P_a(t, T + 1) - \ln P(t, T)].$$

It reflects the non-negativity condition on interest rates at time T , and it holds that $\underline{f}(t, T) \geq 0$ (see Krippner (2012)). Finally, the price of the zero-coupon bond follows from the one-period ZLB-forward rates:

$$\underline{P}(t, T) = e^{-[\underline{f}(t, t) + \dots + \underline{f}(t, T-1)]}.$$

To actually calculate the term structure, note that the auxiliary bond can be interpreted as a long position in the shadow bond and a short position in a call option on that bond with strike price 1:

$$\min\{1, P(T, T + 1)\} = P(T, T + 1) - \max\{P(T, T + 1) - 1, 0\}.$$

Its price is thus

$$P_a(t, T + 1) = P(t, T + 1) - C(t, T, T + 1, 1),$$

where $C(t, T_1, T_2, K)$ is the price of a call option with maturity in T_1 and strike price K on a zero bond with maturity in T_2 . The shadow bond price is given in Equation (3.3), and the option price can be calculated via Fourier inversion. Using the method of Lewis (2000), we get

$$\begin{aligned} & C(t, T, T + 1, 1) \\ &= \frac{1}{2\pi} \int_{iz_i - \infty}^{iz_i + \infty} e^{-izA_{\phi_0}(1,0) + A_{\phi_0}(T-t, -izA_{\phi_1}(1,0)) + A_{\phi_1}(T-t, -izA_{\phi_1}(1,0))' Y_t} w(z) dz, \end{aligned}$$

where z is complex with imaginary part $\Im(z) > 1$, and $w(z) = \frac{-1}{z^2 - iz}$. $A_{\phi_0}(\cdot, \cdot)$ and $A_{\phi_1}(\cdot, \cdot)$ denote the coefficient functions for the pricing function, and Y is a vector of the state variables. Details are given in Appendix A.2. Notably, due the presence of the option component, the ZLB bond yield is no longer linear in the state variables, and need to be computed numerically for given parameters and states of the model.

4 Model Estimation

In our empirical implementation, we first estimate a term structure model using pre-2008 data. In this period nominal rates are sufficiently above a zero lower bound. Quantitatively, the impact of a zero lower bound on yield solutions is quite small, and we can just rely on a linear model without ZLB which is significantly easier to estimate than the full model with a zero lower bound imposed.⁴ Using the estimated model parameters, we can then filter out estimated states and evaluate model-implied yields for the post-2008 period. We do this both under the No-ZLB and the ZLB model for the term structure. This gives rise to the No-ZLB state and yields, and the ZLB states and the model-implied ZLB yields.

To estimate the model in the data we use the per capita series of monthly real consumption expenditure on nondurables and services from the NIPA tables available from the Bureau of Economic Analysis. Monthly inflation represents the log difference of the consumer core price index (CPI). Monthly observations of U.S. Treasury bills and bonds with maturities at three, six months, one to five years, seven years, and ten years are from the Center for Research in Security Prices. The time series span of the monthly data is 1987:M1 to 2014:M12.

4.1 Bayesian Inference

In our empirical analysis we adopt the econometric approach proposed by Schorfheide, Song, and Yaron (2013). It is convenient to cast the model into a state-space form, which consists of a measurement equation that relates the observables to model state variables and a transition equation that describes the law of motion of the state variables.

The measurement of the observed variables is not perfect but subject to measurement noise. Specifically, we entertain measurement errors in the levels of consumption expenditures and the levels of the yields. Let z_t the vector of observables which contains macroeconomic and term structure data and s_t the vector of current and lagged state variables characterizing the level of fundamentals and the measurement errors for consumption. The vector s_t' comprises volatilities of fundamentals. Finally, u_t is a vector of i.i.d. shocks containing the measurement errors for bond yields and consumption. In the case of the

⁴The estimation of the full ZLB model is computationally infeasible because the particle filtering step in the non-linear ZLB model is extremely time-consuming. This estimation strategy is similar to Bauer and Rudebusch (2015).

model without ZLB, the measurement equation is linear, and takes the form

$$z_t = \mu + Zs_t + Z^\nu s_t^\nu + u_t, \quad u_t \sim N(0, \Sigma). \quad (4.1)$$

For the ZLB model, the yields are non-linear functions of the underlying states, i.e., the measurement equation is non-linear and can be represented as:

$$z_t = F(s_t, s_t^\nu) + u_t, \quad u_t \sim N(0, \Sigma), \quad (4.2)$$

where F is a vector-valued function with non-linear components representing the non-linear dependence of the ZLB bond prices on the state variables.

The state transition equations take the the form

$$s_t = \Phi s_{t-1} + w_t(s_{t-1}^\nu) \quad (4.3)$$

$$s_t^\nu = \Phi^\nu s_{t-1}^\nu + w_t^\nu, \quad w_t^\nu \sim \text{Gamma}(k, \theta) \quad (4.4)$$

where $w_t(\cdot)$ is a Gaussian innovation process with a variance that is a function of the volatility process s_t^ν , and w_t^ν is the innovation of the stochastic volatility process which follows a gamma distribution. Further details on the measurement and transition equations are provided in Appendix B.

The model parameters are collected in the vectors

$$\Theta_{\text{preference}} = (\beta, \gamma, \psi)$$

$$\Theta_{\text{dynamics}} = (\mu_c, \mu_\pi, \sigma_c, \sigma_\pi, \rho_{xc}, \rho_{xc,x\pi}, \rho_{x\pi}, \bar{\nu}_{xc}, \rho_{\nu xc}, \sigma_{\nu xc}, \bar{\nu}_{x\pi}, \rho_{\nu x\pi}, \sigma_{\nu x\pi})$$

$$\Theta_{\text{errors}} = (\sigma_{\eta,3m}, \sigma_{\eta,6m}, \sigma_{\eta,1y}, \sigma_{\eta,2y}, \sigma_{\eta,3y}, \sigma_{\eta,4y}, \sigma_{\epsilon,5y}, \sigma_{\eta,7y}, \sigma_{\eta,10y}, \sigma_{\eta,z}).$$

We use a Bayesian approach to make joint inference about $\Theta = (\Theta_{\text{preference}}, \Theta_{\text{dynamics}}, \Theta_{\text{errors}})$ and the latent state vector s and s^ν in Equations (4.3) and (4.4). Bayesian inference requires the specification of a prior distribution $p(\Theta)$ and the evaluation of the likelihood function $p(Y|\Theta)$. We use MCMC methods to generate a sequence of draws $\{\Theta^{(j)}\}_{j=1}^{n_{\text{sim}}}$ from the posterior distribution $p(\Theta|Y) = \frac{p(Y|\Theta)p(\Theta)}{p(Y)}$. The numerical evaluation of the prior density and the likelihood function $p(Y|\Theta)$ is done with a particle filter.

The key aspect of the state-space form of the No ZLB model is that it is linear and Gaussian conditional on the volatility states s_t^ν . We employ an efficient estimation procedure that uses Rao-Blackwellisation to analytically integrate out the state variable s_t . For

details see the references in Schorfheide *et al.* (2013).

4.2 Estimation Results

The 5th and 95th percentiles for the prior distributions for all the model parameters are reported in the left-side columns of Table 1. We generally choose fairly loose priors which cover a wide range of economically plausible parameters to maximize learning from the data. The prior for the subjective discount rate δ captures reasonable values for the real risk-free rate. We choose a loose prior for the IES parameter ψ with a 5%-95% range of 0.3 to 3.5. Notably, the prior distribution allows the IES to be below and above one, and we let the asset-pricing data determine its magnitude. We set the 90% prior credible interval for the risk aversion coefficient γ to range from 5 to 11.

The prior 90% credible intervals for average annualized consumption and inflation range from -7% to 7%. σ_c and σ_π are the average standard deviation of the i.i.d. component of consumption growth and inflation whose 90% prior intervals range from 0.1% to 1.7% at an annualized rate. The prior 90% credible intervals for the persistence of the expected growth and inflation $\rho_{xc}, \rho_{x\pi}$ range from 0.1 to 0.999, and thus accommodate both close-to-iid and very persistent dynamics. We truncate the prior distribution below 1 for stationarity. The prior for the parameter that captures intertemporal feedback between expected growth and inflation $\rho_{xc,x\pi}$ follows a Normal distribution centered at zero with a standard deviation of 0.01. So while expected consumption growth and expected inflation are *a priori* believed to be independent, the 90% credible interval for the corresponding parameter ranges from -0.02 to 0.02. Our choice for the prior reflects the fact that, e.g., Bansal and Shaliastovich (2013) and Piazzesi and Schneider (2006) find that expected inflation and growth tend to be negatively correlated over most of the usually employed sample periods, while the evidence provided by Song (2014) suggests that $\rho_{xc,x\pi}$ can also be positive, e.g., over the last decade.

The prior interval for the persistence of the inflation volatility process $\rho_{\nu x\pi}$ ranges from 0.1 to 0.999. The prior for the unconditional mean of the volatility processes of expected growth and inflation $\bar{\nu}_{xc}, \bar{\nu}_{x\pi}$ implies that its relative magnitude ranges from 2.2% to 14.3% of consumption growth innovations. The prior 90% credible interval for the standard deviation of the volatility process $\sigma_{\nu x\pi}$ is chosen so that the moment generating function of volatility shocks exists.

The median as well as the 5th and 95th percentiles for the posterior distribution are also

reported in Table 1. We fix several macroeconomic parameters to facilitate the identification of the other parameters. In particular, we set the unconditional means of consumption growth and inflation equal to their sample averages. To stabilize the estimates of the expected inflation state in the data, we also fix the volatility of the i.i.d. inflation shocks to about one third of the total inflation variation.

In our estimation, we find that discount rate δ and IES parameter ψ are relatively well identified from the bond yield data. Specifically, our estimate of IES is 1.58, and its 90% credible set is above one. The estimated coefficient of risk aversion γ is 7.7. We find that it is hard to precisely estimate this parameter, and its posterior credible set ranges from 4.6 to 10.3, similar to the prior distribution.

The posterior estimate for the standard deviation of consumption growth is consistent with its standard deviation in the sample. The posterior medians of ρ_{xc} and $\rho_{x\pi}$ are 0.9965 and 0.995, respectively, which imply a high persistence for expected consumption and expected inflation shocks. The posterior median of $\rho_{xc,x\pi}$, which captures the intertemporal feedback between expected growth and inflation, is small but significantly negative -0.0055. This finding is consistent with the evidence in Bansal and Shaliastovich (2013) and Piazzesi and Schneider (2006) that expected inflation predicts a decline in future real growth. Finally, we find that expected inflation volatility is quite persistent and volatile. The level of the volatility is about a third of the volatility of the i.i.d. inflation shocks.

5 Model Implications

5.1 No ZLB Model

We first consider the implications of the No ZLB model. As discussed above this is the model where the zero lower bound for interest rates is not taken into account and bond prices are computed as in Equation (3.3). The blue dashed lines in Figure 1 show real expected growth, expected inflation, and the volatility of the expected inflation inferred from these No ZLB bond prices. Consistent with the parameter estimates in Table 1, the economic states are very persistent and fluctuate significantly over time. Generally, the expected real growth and inflation states track well the low frequency movements in the realized macroeconomic variables. The expected real growth goes down in bad economic times, and it significantly drops in the Great Recession. Expected inflation gradually decreases over the sample. Finally, the volatility of inflation moves considerably over time.

It increases during and right after recessions, and in particular, it exhibits a significant upward jump right around the last recession.

The top panel of Table 2 shows the key moments of nominal yields from 3 month to 10 years to maturity in the data and in the model. The model matches quite well the unconditional level, volatility, and persistence of bond yields in the overall sample. In the data, the term structure is upward sloping: the average yields increase from 3.44% at 3 month to 5.51% at 10 years, and the model produces a similar increase from 3.51% to 5.43%, respectively. The model also captures very well the volatilities of yields: the volatility of short rates is 2.47% in the data relative to 2.48% in the model, and the volatilities decline with the maturity of the bond both in the data and in the model. Yields are very persistent both in the data and in the model.

Figure 2 shows the time-series of yields in the data and in the model. Over most of the sample, the model-implied yields very closely track the observed yields in the data. Naturally, the most noticeable discrepancy between the model-implied and observed yields occurs in the recent sample where the short rates in the data are close to a zero lower bound. To help illustrate the difference between the model-implied and observed yields, we zoom in on the 2008-2014 period in Figure 3. In the data, the short rates essentially stay at zero over this sub-sample. In the model, however, the implied short rates fluctuate significantly between -1% and 0.5%, and are often negative. The discrepancy between the model and the data decreases with the maturity of the yields. Long-term bond yields of 5 years or more are well above zero in the data, and the model tracks the movements in them quite well.

The bottom panel of Table 2 summarizes the key moments of yields in the ZLB periods in the data and the model. In the data, the ZLB period corresponds to the 2008-2014 sub-sample, while in the model we identify these periods when the 1 month short rate falls below zero. Consistent with the sub-sample evidence in the figure, the No ZLB model implies negative and volatile bond yields at short maturities, while in the data the short rates stay close to zero. For a 3-month bond, its average level is 0.27% in the data relative to -1.19% in the model. The difference between the model and data yields shrinks at longer maturities. In a ZLB period, a 10 year yield is 3.01% in the data, and it is 2.60% in the model.

To further help assess the frequency and significance of negative short rates in the model, we show the unconditional distribution of model-implied yields in the top panel of Figure 4. In our benchmark estimation, the 1-month nominal yield has a 12% unconditional

probability of falling below zero. This probability decreases with the maturity of the yield. It is equal to about 5% for a 5-year yield, and is essentially zero for a 10-year rate. The bottom panel of the figure shows the distribution of the duration of the negative rate. Due to a high persistence of the states, the periods of negative rates are quite long-lasting. On average, the short rate is expected to stay negative for 11 months, but as shown in the Figure, the model assigns non-negligible probabilities for the durations of several years.

The above discussion has focused on unconditional properties of yields. There are further important conditional aspects to the dynamics of the bond prices and the implications for the ZLB periods. As discussed above, nominal yields are linear functions of the expected real growth, expected inflation, and inflation volatility states in the No ZLB model. We show the linear loadings of yields on these three factors on Figure 5. Consistent with the theoretical discussion in Section 2 and in Bansal and Shaliastovich (2013), an increase in expected real growth or expected inflation increases nominal yields at all maturities. High inflation volatility has a negative impact on short-term yields due to a precautionary savings motive, and increases long-term yields due to a rise in the inflation risk premium. Note that for No ZLB model, the three states linearly explain the variation in yields, so the R^2 in the projection is equal to one.

The effect of the states on short term yields determines their impact on the likelihood and the duration of the ZLB (negative interest rate) periods. Specifically, an increase in expected real growth, an increase in expected inflation or a decrease in inflation volatility makes it less likely for the short rates to become negative, and thus decreases the probability of getting into a ZLB region. The most important drivers of the yields and thus of the occurrence of ZLB periods are news about expected inflation and expected growth. Quantitatively, using a probit regression, we find that a one-standard deviation increase in either the expected inflation or expected real growth decreases the likelihood of observing negative short rates over the next year by about 1%.

5.2 ZLB Model

In this section we evaluate the economic implications of the ZLB restriction. We fix the parameter estimates from the benchmark model, and filter out the economic states using the ZLB model solution for the nominal yields.

Figure 1 shows the time-series of the No ZLB and ZLB implied states. As to be expected, the corresponding states are essentially identical away from the ZLB period in the data.

In these regions the value of the ZLB-related optionality is quite small, so that the two models are quantitatively similar, which leads to the same values for the states. The states, however, are quite different in the 2008-2014 period.

To help assess the differences, we show the time-series of the states over the recent period in Figure 6. The introduction of the ZLB restriction has the most significant implications for the implied inflation dynamics. Expected inflation drops, while inflation uncertainty increases relative to the No ZLB models. These effects can be quite significant. The gap between the filtered expected inflation states can be as large as 1% per year. Relative to the No ZLB model, the volatility of expected inflation increases by about a quarter in 2010 from 0.08 to 0.10, and stays above the estimate from the No ZLB model until the end of the sample.

The two panels of Table 2 summarize the unconditional implications of the ZLB relative to the No ZLB model. In the population, the ZLB yields are always larger than their No ZLB counterparts. The largest difference of 11 basis points is for the shortest-maturity rates (3 months) which are most affected by the ZLB restriction, and it drops to 3 basis points for 10 year yields.⁵ The ZLB rates are also less volatile relative to No ZLB rates as they do not fluctuate below zero. The difference between the two models is especially pronounced during the negative rate periods, as shown in the bottom panel of the table. While the No ZLB rates are negative and volatile in those periods, the ZLB rates are positive and quite stable, as in the data. Overall, the ZLB model quite successfully captures the bond prices in the ZLB periods, and has comparable implications to the No ZLB model when bond yields are away from zero.

Next we consider the model implications for the sample yields. Figure 2 shows the implied yields in the ZLB model over the whole sample, while Figure 3 zooms in on the ZLB period. Away from the ZLB period, the yields under the two models are essentially the same. However, there are noticeable differences between the two models in the ZLB period. While the No ZLB short rates are volatile and negative, the ZLB short rates are basically zero, as in the data. Consistent with the population statistics, the long-term yields from both models are fairly close to each other as the impact of the ZLB optionality diminishes at long maturities.

The ZLB model yields plotted on the Figures are based on the ZLB model solution and the ZLB state variables. That is, the difference between the in-sample ZLB and No

⁵Christensen and Rudebusch (2015b) show qualitatively similar results for a reduced-form term structure model.

ZLB model yields is driven by the difference in the pricing equations and the difference in the state variables. Interestingly, the two channels have the opposite effect on the term structure, as we document in Figure 7. Here we plot the model-implied yields for four different specifications of our model with respect to the state variables (inferred from either the ZLB or the No ZLB model) and the way to compute bond prices (according to the ZLB or the No ZLB model). Switching from the No ZLB to the ZLB state variables lowers the overall level of yields and increases the slope of the term structure. This is a direct consequence of lower expected inflation and higher inflation volatility states implied by the ZLB model. Lower expected inflation brings down the level of yields almost uniformly across maturities, while higher inflation volatility decreases short-term and increases long-term yields (see Figure 5).

In contrast, switching from the No ZLB to the ZLB model solution for fixed states increases the level and dampens the slope of the term structure. This follows nearly mechanically as the No ZLB short rates, unlike the long rates, are negative over the sample, so incorporating the ZLB restriction lifts the short end of the term structure to zero and also has a positive, albeit smaller effect on long rates. In our estimation, the differences in model and state effects nearly offset each other for long yields, so the No ZLB long-term yields evaluated at the No ZLB state variables are quite close to the ZLB long-term rates evaluated at the ZLB state variables. For short rates, incorporating the ZLB restriction plays a far more important role than the difference in the state variables. Indeed, because the No ZLB rates are mostly below zero, incorporating the ZLB restriction essentially makes the short rates to be zero under both the No ZLB and the ZLB states.

Next we consider the conditional implications of the ZLB model. The No ZLB model is an affine term structure model, so the associated interest rates are linear functions of the economic states. The ZLB model incorporates a restriction, which we represent via a non-linear optionality, so that rates are no longer affine in the states. Figure 5 contrasts the linear dependencies of nominal yields on the states across the two models. The left column of graphs shows the slope coefficients in the regressions of bond yields on the three states from a long unconditional simulation of the model. On average, the No ZLB yields are positive away from the ZLB restriction, so in the steady state the two models are quite similar, since the unconditional probability of a ZLB period occurring is low. The bond loadings are close to each other across the two models, and the R^2 s in linear regressions are close to the ones for the ZLB model.

The two models, however, behave very differently in the ZLB periods (periods of neg-

ative short rates), as shown by the graphs in the right column of the figure. Naturally, bond loadings do not change for the No ZLB model, but they are quite different for the ZLB model. Indeed, in the ZLB model the short rates are zero in the ZLB period, so they do not fluctuate with economic states. Hence, the ZLB bond sensitivities to the states in ZLB periods start at zero, and then gradually increase to the No ZLB levels. Notably, the non-linearities induced by the ZLB model solution are quite sizeable for short-term bonds. The R^2 in linear regression is below 60% for 1 year yield, and it increases to 90% for at 5-year maturities.

The non-linearity of the bond yields has further implications for the bond risk premia. In the benchmark No ZLB model, the risk premia are linear in the volatility of expected inflation. Indeed, as shown in Figure 8, the bond risk premia loadings on expected growth and expected inflation are zero, and they are positive and increasing for the inflation volatility. Unconditionally, the ZLB bond risk premia are similar to their No ZLB counterparts, although one can already see differences regarding the impact of x_c and x_π , which is by definition equal to zero in the No ZLB model.

There are much larger discrepancies, however, between the two models during the ZLB periods. As shown on the right column of graphs, the ZLB bond risk premia in ZLB periods are positively and significantly affected by the expected growth and expected inflation, while the impact of the volatility of expected inflation somewhat diminishes relative to the No ZLB case. Relative to the No ZLB period, the ZLB short rates are fixed at zero in ZLB periods. Hence, all the positive shocks to the No ZLB short rates due to an increase in expected growth, expected inflation, or a decrease in the volatility of expected inflation now manifest as movements in the ZLB risk premia.

We show the time-series of model-implied bond risk premia in Figure 9. The implied risk premia generally track movements in the volatility of expected inflation, as shown in Figure 6. However, there are important differences between the premia implied by the two models. As for the levels of the yields, bond risk premia are affected by the No ZLB versus the ZLB state variables and the No ZLB versus the ZLB bond price solutions, which have opposite effects on the levels of the risk premia. Switching to the ZLB states generally increases risk premia due to a higher level of the volatility of expected inflation inferred in the ZLB model. On the other hand, incorporating the ZLB restriction generally decreases the level of the premia as it increases the short rates and dampens the slope of the term structure.

5.3 Other Model Implications

In this section we discuss additional model implications for the shadow rates, lift-off probabilities, and yield forecasts.

Shadow Rates. The implied ‘shadow’ interest rates correspond to the No ZLB rates in Equation (3.4) evaluated at the ZLB state variables. These are the rates which would prevail in the absence of the ZLB restriction, given the (true, i.e., the ZLB-based) economic states. Analogously, the No ZLB model risk premia evaluated at the ZLB state variables are called shadow risk premia.

The shadow rates are shown by the dashed-dotted lines in Figure 7. They are quite large and negative, and very volatile. On average, the 3-month shadow rate is equal to -0.88% and its volatility is 1.64%. The shadow short rates stay at low levels for a protracted period from mid 2011 to 2013, and they gradually increase to zero by the end of the sample. We also plot a shadow bond risk premium in Figure 9. Over the period of 2009 to 2012, the shadow bond risk premium is about a quarter larger than the risk premia implied by the ZLB model, and the risk premia which are based on the No-ZLB states. This highlights the importance of a zero lower bound for a correct measurement and identification of the risk premia in financial markets.

Lift-off probabilities. The lift-off probabilities determine the likelihood that interest rates become positive within a predetermined horizon. We compute the time-series of the conditional lift-off probabilities for yield maturities of 3 months to 5 years, and horizons of 1 month, 1 year, and 5 years. Figure 10 depicts these probabilities computed using the ZLB states, while Figure 11 shows the probabilities under the No-ZLB states. As shown in Figure 10, for the most of the 2010-2014 period the ZLB state-based model assigns almost a zero chance that short, 3-month rates would become positive within one month. The one-year lift-off probabilities fluctuate between 20% in 2012 and almost 80% in the beginning of 2011 and after the mid 2013. The 5-year lift-off probabilities are generally above 70% for the whole sample. The lift-off probabilities uniformly increase for long-maturity bonds. However, for the period from mid 2011 to mid 2013, the model assigns substantial probabilities of remaining at a zero lower bound even for long-term rates. For example, the probability that one-year yields cross zero lower bound within 5 years is estimated to be about 80% in mid 2011 to 2013 period. For 5-year yields, the probability of a lift-off within a month is as low as 20% in mid 2012.

We compare these findings to the model which relies on No ZLB extracted states. As

shown in Figure 11, the lift-off probabilities are substantially higher in the No ZLB model. Indeed, the 1-year lift-off probabilities are generally above 50% for the 3-month rate, and are close to 100% for the 1-year yield at the 5-year horizons. The yields based on the No ZLB states for maturities of 3 years and longer are above zero lower bound, so the lift-off probabilities are by definition equal to 100%.

Yield forecasts. Figure 12 considers model forecasts of short- (3 month) and long-maturity (10 year) bond yields, conditional on the information set at the end of the sample.⁶ We further compare the model forecasts to the 1- to 4-quarter ahead yield forecasts from the Survey of Professional Forecasters. The yield forecasts increase both in the model and in the data. The increase is steeper in the data for short rates. The short rate forecasts three and four quarters ahead are somewhat above the model predictions; however, all the data forecasts are well within the confidence bands generated by the model. The forecasts for the 10-year yield are actually quite close in the data and in the model. Notably, the model implies a rather slow recovery for the interest rates. Indeed, by 2025, the median short and long rates are projected to be about 1% and 3.8%, well below their unconditional values of 3.6% and 5.5%. The model further assigns only a 5% chance that the economy reaches the steady-state values for the interest rates by 2020. The uncertainty about future rates in the model is also quite considerable. By 2020, both the unconditional levels and values of zero are contained in the model 5-95% confidence set.

6 Conclusion

Interest rates have been at historically low levels since the beginning of the financial crisis. The fact that interest rates cannot decrease any more when the zero lower bound restriction is binding has implications for the level and dynamics of interest rates and for the information extracted from the term structure.

We study the implications of the zero lower bound on nominal interest rates in a long-run risks model with expected real growth, expected inflation and inflation volatility. The models with and without the zero lower bound give similar results away from the ZLB region, but differ substantially once the restriction becomes binding. Compared to the standard No ZLB model, interest rates are higher and the slope of the term structure is dampened in the ZLB model. Due to the higher short rate, risk premia decrease.

⁶Bauer and Rudebusch (2015) and Christensen, Lopez, and Rudebusch (2015) forecast yields in the context of a reduced-form term structure model, which obeys the ZLB.

The introduction of the zero lower bound restriction also changes the dependence of interest rates and risk premia on the state variables. In particular, in the ZLB region interest rates are less sensitive to changes in the state variables than in a model without the ZLB restriction. Risk premia, on the other hand, are not only driven by inflation volatility as in the No ZLB model, but now they also depend on expected real growth and expected inflation.

We estimate the model using Bayesian MCMC method. In the periods in which yields are close to zero, the ZLB model gives substantially different estimates for the state variables compared to the standard model. The model-implied estimates of expected inflation are lower under the ZLB model, while the estimated inflation volatility is higher. The observed low volatility of short-term rates near the zero lower bound is thus not due to low inflation volatility, but due to the fact that rates cannot fully react to changing economic conditions anymore once they are close to zero.

Overall, these results show that incorporating the ZLB feature into an equilibrium model is an important step forward compared to standard affine models.

A Asset Pricing

A.1 Equilibrium Stochastic Discount Factor

To determine the equilibrium stochastic discount factor, we follow Eraker and Shaliastovich (2008) and Bansal and Shaliastovich (2013). With Epstein-Zin preferences, the logarithm of the stochastic discount factor is

$$m_{t+1} = \theta \ln \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1 - \theta) r_{c,t+1} \quad (\text{A.1})$$

where $\theta = \frac{1-\gamma}{1-\frac{\gamma}{\psi}}$ and r_c is the logarithm of the return on the consumption claim. The Campbell-Shiller approximation for r_c is

$$r_{c,t+1} = \kappa_0 + \kappa_1 w c_{t+1} - w c_t + \Delta c_{t+1}.$$

wc is the logarithm of the wealth-consumption ratio, and the linearizing coefficients κ_0 and κ_1 are

$$\begin{aligned} \kappa_1 &= \frac{e^{\overline{wc}}}{1 + e^{\overline{wc}}} \\ \kappa_0 &= -\ln \kappa_1 + (1 - \kappa_1) \overline{wc}. \end{aligned}$$

They depend on the mean \overline{wc} of the log wealth-consumption ratio. The usual affine guess for the log wealth-consumption ratio is

$$w c_t = A_0 + A_{xc} x_{c,t} + A_{x\pi} x_{\pi,t} + A_{\nu xc} \nu_{xc,t} + A_{\nu x\pi} \nu_{x\pi,t}.$$

Plugging the approximated return on the consumption claim and this guess into the Euler equation for the consumption claim

$$\mathbf{E}_t [e^{m_{t+1} + r_{c,t+1}}] = 1$$

gives the coefficients

$$\begin{aligned} A_{xc} &= \left(1 - \frac{1}{\psi}\right) \frac{1}{1 - \kappa_1 \rho_{xc}} \\ A_{x\pi} &= \left(1 - \frac{1}{\psi}\right) \frac{\kappa_1 \rho_{xc, x\pi}}{(1 - \kappa_1 \rho_{xc})(1 - \kappa_1 \rho_{x\pi})} \end{aligned}$$

and

$$\begin{aligned} A_{\nu xc} &= \frac{(1 - \gamma) \left(1 - \frac{1}{\psi}\right) \sigma_{xc}^2}{2(1 - \kappa_1 \rho_{\nu xc})} \left(\frac{\kappa_1}{1 - \kappa_1 \rho_{xc}}\right)^2 \\ A_{\nu x\pi} &= \frac{(1 - \gamma) \left(1 - \frac{1}{\psi}\right) \sigma_{x\pi}^2}{2(1 - \kappa_1 \rho_{\nu x\pi})} \left(\frac{\kappa_1^2 \rho_{xc, x\pi}}{(1 - \kappa_1 \rho_{xc})(1 - \kappa_1 \rho_{x\pi})}\right)^2. \end{aligned}$$

The constant A_0 is

$$A_0 = \ln \frac{\kappa_1}{1 - \kappa_1} - A_{\nu xc} \frac{k_{\nu xc} \theta_{\nu xc}}{1 - \rho_{\nu xc}} - A_{\nu x\pi} \frac{k_{\nu x\pi} \theta_{\nu x\pi}}{1 - \rho_{\nu x\pi}}.$$

The linearizing constants follow from

$$\begin{aligned} \kappa_0 &= -\kappa_1 \ln \kappa_1 - (1 - \kappa_1) \ln(1 - \kappa_1) \\ \ln \kappa_1 &= \ln \beta + A_{\nu xc} (1 - \kappa_1) \frac{k_{\nu xc} \theta_{\nu xc}}{1 - \rho_{\nu xc}} + A_{\nu x\pi} (1 - \kappa_1) \frac{k_{\nu x\pi} \theta_{\nu x\pi}}{1 - \rho_{\nu x\pi}} \\ &\quad + \left(1 - \frac{1}{\psi}\right) \mu_c + 0.5(1 - \gamma) \left(1 - \frac{1}{\psi}\right) \sigma_c^2 \\ &\quad - \frac{1}{\theta} k_{\nu xc} \ln(1 - \theta \kappa_1 A_{\nu xc} \theta_{\nu xc}) - \frac{1}{\theta} k_{\nu x\pi} \ln(1 - \theta \kappa_1 A_{\nu x\pi} \theta_{\nu x\pi}). \end{aligned}$$

Plugging the log wealth-consumption ratio and the dynamics of consumption into Equation (A.1) for the (real) stochastic discount factor gives

$$\begin{aligned} m_{t+1} &= m_0 + m_{xc} x_{c,t} + m_{x\pi} x_{\pi,t} + m_{\nu xc} \nu_{xc,t} + m_{\nu x\pi} \nu_{x\pi,t} \\ &\quad - \lambda_c \sigma_c \varepsilon_{c,t+1} - \lambda_\pi \sigma_\pi \varepsilon_{\pi,t+1} - \lambda_{xc} \sqrt{\nu_{xc,t}} \varepsilon_{xc,t+1} - \lambda_{x\pi} \sqrt{\nu_{x\pi,t}} \varepsilon_{x\pi,t+1} \\ &\quad - \lambda_{\nu xc} \varepsilon_{\nu xc,t+1} - \lambda_{\nu x\pi} \varepsilon_{\nu x\pi,t+1} \end{aligned}$$

where the constant is

$$m_0 = \theta \ln \beta + (1 - \theta) \ln \kappa_1 - \gamma \mu_c - (1 - \theta)(1 - \kappa_1) \left[A_{\nu xc} \frac{k_{\nu xc} \theta_{\nu xc}}{1 - \rho_{\nu xc}} + A_{\nu x\pi} \frac{k_{\nu x\pi} \theta_{\nu x\pi}}{1 - \rho_{\nu x\pi}} \right],$$

and the sensitivities are

$$\begin{aligned} m_{xc} &= -\frac{1}{\psi} & m_{x\pi} &= 0 \\ m_{\nu xc} &= (1 - \theta)(1 - \kappa_1 \rho_{\nu xc}) A_{\nu xc} & m_{\nu x\pi} &= (1 - \theta)(1 - \kappa_1 \rho_{\nu x\pi}) A_{\nu x\pi}, \end{aligned}$$

and the market prices of risk are

$$\begin{aligned} \lambda_c &= \gamma & \lambda_\pi &= 0 \\ \lambda_{xc} &= \kappa_1 (1 - \theta) A_{xc} & \lambda_{x\pi} &= \kappa_1 (1 - \theta) A_{x\pi} \\ \lambda_{\nu xc} &= \kappa_1 (1 - \theta) A_{\nu xc} & \lambda_{\nu x\pi} &= \kappa_1 (1 - \theta) A_{\nu x\pi}. \end{aligned}$$

The nominal stochastic discount factor follows from the real stochastic discount factor and inflation via

$$m_{t+1}^{\$} = m_{t+1} - \pi_{t+1}.$$

This gives

$$\begin{aligned}
m_{t+1}^{\$} &= m_0^{\$} + m_{xc}^{\$}x_{c,t} + m_{x\pi}^{\$}x_{\pi,t} + m_{\nu xc}^{\$}\nu_{xc,t} + m_{\nu x\pi}^{\$}\nu_{x\pi,t} \\
&\quad - \lambda_c^{\$}\sigma_c\varepsilon_{c,t+1} - \lambda_{\pi}^{\$}\sigma_{\pi}\varepsilon_{\pi,t+1} - \lambda_{xc}^{\$}\sqrt{\nu_{xc,t}}\varepsilon_{xc,t+1} - \lambda_{x\pi}^{\$}\sqrt{\nu_{x\pi,t}}\varepsilon_{x\pi,t+1} \\
&\quad - \lambda_{\nu xc}^{\$}\varepsilon_{\nu xc,t+1} - \lambda_{\nu x\pi}^{\$}\varepsilon_{\nu x\pi,t+1},
\end{aligned}$$

where the constant is

$$m_0^{\$} = \theta \ln \beta + (1 - \theta) \ln \kappa_1 - \gamma \mu_c - \mu_{\pi} - (1 - \theta)(1 - \kappa_1) \left[A_{\nu xc} \frac{k_{\nu xc} \theta_{\nu xc}}{1 - \rho_{\nu xc}} + A_{\nu x\pi} \frac{k_{\nu x\pi} \theta_{\nu x\pi}}{1 - \rho_{\nu x\pi}} \right],$$

the sensitivities are

$$\begin{aligned}
m_{xc}^{\$} &= -\frac{1}{\psi} & m_{x\pi}^{\$} &= -1 \\
m_{\nu xc}^{\$} &= (1 - \theta)(1 - \kappa_1 \rho_{\nu xc}) A_{\nu xc} & m_{\nu x\pi}^{\$} &= (1 - \theta)(1 - \kappa_1 \rho_{\nu x\pi}) A_{\nu x\pi},
\end{aligned}$$

and the market prices of risk are

$$\begin{aligned}
\lambda_c^{\$} &= \gamma & \lambda_{\pi}^{\$} &= 1 \\
\lambda_{xc}^{\$} &= \kappa_1(1 - \theta) A_{xc} & \lambda_{x\pi}^{\$} &= \kappa_1(1 - \theta) A_{x\pi} \\
\lambda_{\nu xc}^{\$} &= \kappa_1(1 - \theta) A_{\nu xc} & \lambda_{\nu x\pi}^{\$} &= \kappa_1(1 - \theta) A_{\nu x\pi}.
\end{aligned}$$

A.2 Calculation of Bond Prices

The calculation of the shadow bond prices and the option prices relies on the pricing function ϕ which is defined by

$$\phi(u, t, T, Y_t) = E_t \left[M_{t,T}^{\$} e^{u' Y_T} \right]$$

where $M_{t,T}^{\$}$ is the nominal stochastic discount factor at time t for payments in T , and

$$\begin{aligned}
u &= (u_c, u_{\pi}, u_{xc}, u_{x\pi}, u_{\nu xc}, u_{\nu x\pi})', \\
Y_t &= (c_t, \pi_t, x_{c,t}, x_{\pi,t}, \nu_{xc,t}, \nu_{x\pi,t})'.
\end{aligned}$$

The usual affine guess for the pricing function is

$$\phi(u, t, T, Y_t) = \exp\{A_{\phi 0}(T - t, u) + A_{\phi 1}(T - t, u)' Y_t\}.$$

The coefficient $A_{\phi 1}$ is given by $A_{\phi 1} = (A_{\phi 1,c}, A_{\phi 1,\pi}, A_{\phi 1,xc}, A_{\phi 1,x\pi}, A_{\phi 1,\nu xc}, A_{\phi 1,\nu x\pi})'$. It holds that $A_{\phi 1,c} = A_{\phi 1,\pi} = 0$. The coefficients for the state variables solve a system of ordinary differential

equations:

$$\begin{aligned}
A_{\phi 1,xc}(\tau + 1, u) &= -\frac{1}{\psi} + \rho_{xc}A_{\phi 1,xc}(\tau, u) \\
A_{\phi 1,x\pi}(\tau + 1, u) &= -1 + \rho_{xc,x\pi}A_{\phi 1,xc}(\tau, u) + \rho_{x\pi}A_{\phi 1,x\pi}(\tau, u) \\
A_{\phi 1,\nu xc}(\tau + 1, u) &= (1 - \theta)(1 - \kappa_1\rho_{\nu xc})A_{\nu xc} + \rho_{\nu xc}A_{\phi 1,\nu xc}(\tau, u) \\
&\quad + 0.5(A_{\phi 1,xc}(\tau, u) - \kappa_1(1 - \theta)A_{xc})^2 \\
A_{\phi 1,\nu x\pi}(\tau + 1, u) &= (1 - \theta)(1 - \kappa_1\rho_{\nu x\pi})A_{\nu x\pi} + \rho_{\nu x\pi}A_{\phi 1,\nu x\pi}(\tau, u) \\
&\quad + 0.5(A_{\phi 1,x\pi}(\tau, u) - \kappa_1(1 - \theta)A_{x\pi})^2
\end{aligned}$$

subject to the boundary condition $A_{\phi 1,xc}(0, u) = u_{xc}$, $A_{\phi 1,x\pi}(0, u) = u_{x\pi}$, $A_{\phi 1,\nu xc}(0, u) = u_{\nu xc}$, and $A_{\phi 1,\nu x\pi}(0, u) = u_{\nu x\pi}$. Finally, it holds that

$$\begin{aligned}
A_{\phi 0}(\tau + 1, u) &= \tilde{m}_0^{\$} + A_{\phi 0}(\tau, u) + 0.5\gamma^2\sigma_c^2 + 0.5\sigma_{\pi}^2 \\
&\quad - k_{\nu xc} \ln [1 - (A_{\phi 1,\nu xc}(\tau, u) - \kappa_1(1 - \theta)A_{\nu xc}) \theta_{\nu xc}] \\
&\quad - k_{\nu x\pi} \ln [1 - (A_{\phi 1,\nu x\pi}(\tau, u) - \kappa_1(1 - \theta)A_{\nu x\pi}) \theta_{\nu x\pi}].
\end{aligned}$$

The shadow bond prices are

$$P(t, T, Y_t) = e^{A_{\phi 0}(T-t,0) + A_{\phi 1}(T-t,0)'Y_t},$$

i.e. we have $B_0(T - t) = A_{\phi 0}(T - t, 0)$ and $B(T - t) = A_{\phi 1}(T - t, 0)$.

Option prices can be calculated following Lewis (2000). The price of a call option with strike price K and maturity in T on the zero-coupon bond with maturity in $T + 1$ is

$$\begin{aligned}
C(t, T, T + 1, K) &= \mathbb{E}_t \left[M_{t,T}^{\$} \max\{P(T, T + 1) - K, 0\} \right] \\
&= \frac{1}{2\pi} \mathbb{E}_t \left[\int_{iz_i - \infty}^{iz_i + \infty} M_{t,T}^{\$} e^{-iz \ln P(T, T + 1)} w(z) dz \right] \\
&= \frac{1}{2\pi} \mathbb{E}_t \left[\int_{iz_i - \infty}^{iz_i + \infty} M_{t,T}^{\$} e^{-iz [A_{\phi 0}(1,0) + A_{\phi 1}(1,0)'Y_T]} w(z) dz \right] \\
&= \frac{1}{2\pi} \int_{iz_i - \infty}^{iz_i + \infty} e^{-iz A_{\phi 0}(1,0) + A_{\phi 0}(T-t, -iz A_{\phi 1}(1,0)) + A_{\phi 1}(T-t, -iz A_{\phi 1}(1,0))'Y_t} w(z) dz
\end{aligned}$$

where $w(z) = \frac{-K^{iz+1}}{z^2 - iz}$ and $Im(z) > 1$.

The price of the shadow bond is thus given by

$$\begin{aligned}
P_a(t, T) &= e^{A_{\phi 0}(T-t,0) + A_{\phi 1}(T-t,0)'Y_t} \\
&\quad - \frac{1}{2\pi} \int_{iz_i - \infty}^{iz_i + \infty} e^{-iz A_{\phi 0}(1,0) + A_{\phi 0}(T-t-1, -iz A_{\phi 1}(1,0)) + A_{\phi 1}(\tau, -iz A_{\phi 1}(1,0))'Y_t} w(z) dz.
\end{aligned}$$

B State-Space Representation

Under No-ZLB model specification, the state-space representation of the model is linear:

$$\begin{aligned} z_t &= \mu + Zs_t + Z^\nu s_t^\nu + u_t, \quad u_t \sim N(0, \Sigma), \\ s_t &= \Phi s_{t-1} + w_t(s_{t-1}^\nu), \quad s_t^\nu = \Phi^\nu s_{t-1}^\nu + w_t^\nu, \quad w_t^\nu \sim \text{Gamma}(k, \theta). \end{aligned} \quad (\text{B.1})$$

Measurement Equation: $z_t = \mu + Zs_t + Z^\nu s_t^\nu + u_t$.

$$\begin{aligned} \begin{bmatrix} \Delta c_{t+1} \\ \pi_{t+1} \\ z_{t+1} \\ \vdots \\ y_{j,t+1} \\ \vdots \end{bmatrix} &= \begin{bmatrix} \mu_c \\ \mu_\pi \\ 0 \\ \vdots \\ \beta_j^0 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_j^c & 0 & \beta_j^\pi & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_{c,t+1} \\ x_{c,t} \\ x_{\pi,t+1} \\ x_{\pi,t} \\ \sigma_{\eta,c}\eta_{c,t+1} \\ \sigma_{\eta,c}\eta_{c,t} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \beta_j^{\nu,c} & \beta_j^{\nu,\pi} \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \nu_{xc,t+1} \\ \nu_{x\pi,t+1} \end{bmatrix} + \begin{bmatrix} \sigma_c \epsilon_{c,t+1} \\ \sigma_\pi \epsilon_{\pi,t+1} \\ \sigma_z \epsilon_{z,t+1} \\ \vdots \\ \sigma_{\eta,j} \eta_{j,t+1} \\ \vdots \end{bmatrix}, \quad j \in \{3m, 6m, 1y-5y, 7y, 10y\}. \end{aligned}$$

Transition Equations: $s_t = \Phi s_{t-1} + w_t(s_{t-1}^\nu), \quad s_t^\nu = \Phi^\nu s_{t-1}^\nu + w_t^\nu$

$$\begin{aligned} \begin{bmatrix} x_{c,t+1} \\ x_{c,t} \\ x_{\pi,t+1} \\ x_{\pi,t} \\ \sigma_{\eta,c}\eta_{c,t+1} \\ \sigma_{\eta,c}\eta_{c,t} \end{bmatrix} &= \begin{bmatrix} \rho_{xc} & 0 & \rho_{xc,x\pi} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{x\pi} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{c,t} \\ x_{c,t-1} \\ x_{\pi,t} \\ x_{\pi,t-1} \\ \sigma_{\eta,c}\eta_{c,t} \\ \sigma_{\eta,c}\eta_{c,t-1} \end{bmatrix} + \begin{bmatrix} \sqrt{\nu_{xc,t}} \epsilon_{xc,t+1} \\ 0 \\ \sqrt{\nu_{x\pi,t}} \epsilon_{x\pi,t+1} \\ 0 \\ \sigma_{\eta,c}\eta_{c,t+1} \\ 0 \end{bmatrix} \\ \begin{bmatrix} \nu_{xc,t+1} \\ \nu_{x\pi,t+1} \end{bmatrix} &= \begin{bmatrix} \rho_{\nu xc} & 0 \\ 0 & \rho_{\nu x\pi} \end{bmatrix} \begin{bmatrix} \nu_{xc,t} \\ \nu_{x\pi,t} \end{bmatrix} + \begin{bmatrix} w_{\nu xc,t+1} \\ w_{\nu x\pi,t+1} \end{bmatrix} \end{aligned}$$

where $\epsilon_{c,t}, \epsilon_{\pi,t}, \epsilon_{z,t}, \epsilon_{xc,t}, \epsilon_{x\pi,t}, \eta_{c,t}, \eta_{j,t} \sim N(0, 1)$ and $w_{\nu xi,t} \sim \text{Gamma}(k_i, \theta_i)$ for $i = c, \pi$ where $k_i = \frac{(1-\rho_{\nu xi})^2 \bar{\nu}_{xi}^2}{\sigma_{\nu xi}^2}$ and $\theta_i = \frac{\sigma_{\nu xi}^2}{(1-\rho_{\nu xi})\bar{\nu}_{xi}}$.

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Tables and Figures

Table 1: Model Parameter Estimates

	Prior		Posterior		
	5%	95%	median	5%	95%
δ	[0.9950	0.9999]	0.9985	[0.9981	0.9992]
γ	[5.00	11.00]	7.66	[4.63	10.25]
ψ	[0.31	3.45]	1.58	[1.23	1.96]
$\mu_c \times 1200$	—	—	1.56	—	—
$\sigma_c \times 100\sqrt{12}$	[0.12	1.67]	0.76	[0.73	0.90]
$\mu_\pi \times 1200$	—	—	2.76	—	—
$\sigma_\pi \times 100\sqrt{12}$	—	—	0.24	—	—
ρ_{xc}	[0.0800	0.9999]	0.9965	[0.9933	0.9984]
$\rho_{xc,x\pi}$	[-0.0164	0.0164]	-0.0055	[-0.0009	-0.0029]
$\rho_{x\pi}$	[0.0800	0.9999]	0.9950	[0.9893	0.9986]
$\sqrt{12\bar{\nu}_{xc}} \times 100$	[0.0172	0.1105]	0.0380	[0.0313	0.0412]
$\sqrt{12\bar{\nu}_{x\pi}} \times 100$	[0.0172	0.1105]	0.0840	[0.0571	0.1350]
$\rho_{\nu x\pi}$	[0.0800	0.9999]	0.9493	[0.9124	0.9674]
$\sqrt{12\sigma_{\nu x\pi}} \times 100$	[0.0197	0.0469]	0.0321	[0.0149	0.0481]

The table shows the prior and posterior distributions of the parameters of the model. ‘—’ indicates that the parameter value is fixed.

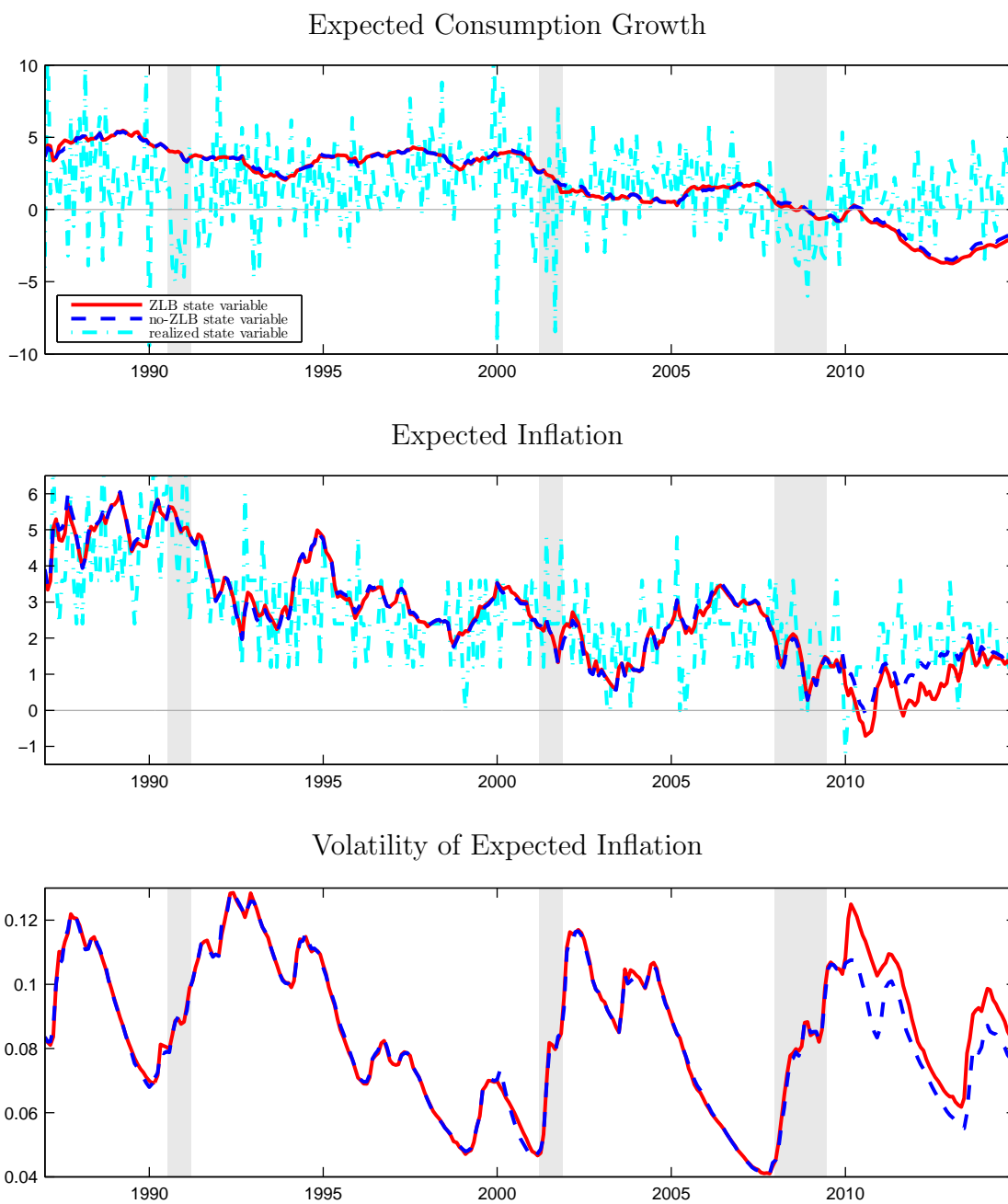
Table 2: Nominal Yields: Data and Models

Panel A: Unconditional Bond Yields									
	Data			No ZLB			ZLB		
	mean	std	AR(1)	mean	std	AR(1)	mean	std	AR(1)
3m	3.44	2.47	0.999	3.51	2.48	0.990	3.62	2.29	0.990
6m	3.69	2.60	0.999	3.56	2.45	0.991	3.67	2.26	0.990
1y	3.86	2.62	0.997	3.68	2.38	0.991	3.77	2.20	0.991
3y	4.36	2.50	0.995	4.15	2.17	0.993	4.22	2.05	0.992
5y	4.78	2.32	0.994	4.59	2.01	0.993	4.64	1.92	0.993
7y	5.12	2.17	0.993	4.97	1.86	0.993	5.01	1.79	0.993
10y	5.51	2.00	0.993	5.43	1.67	0.993	5.46	1.62	0.993

Panel B: Bond Yields in ZLB periods									
	Data			No ZLB			ZLB		
	mean	std	AR(1)	mean	std	AR(1)	mean	std	AR(1)
3m	0.27	0.54	0.847	-1.19	1.11	0.892	0.03	0.05	0.176
6m	0.36	0.59	0.858	-1.05	1.09	0.896	0.09	0.12	0.424
1y	0.47	0.56	0.850	-0.78	1.08	0.906	0.22	0.24	0.687
3y	1.03	0.63	0.888	0.20	1.10	0.934	0.85	0.58	0.902
5y	1.73	0.73	0.912	1.02	1.11	0.948	1.49	0.74	0.937
7y	2.34	0.80	0.922	1.73	1.10	0.956	2.09	0.82	0.950
10y	3.01	0.84	0.928	2.60	1.07	0.964	2.86	0.87	0.962

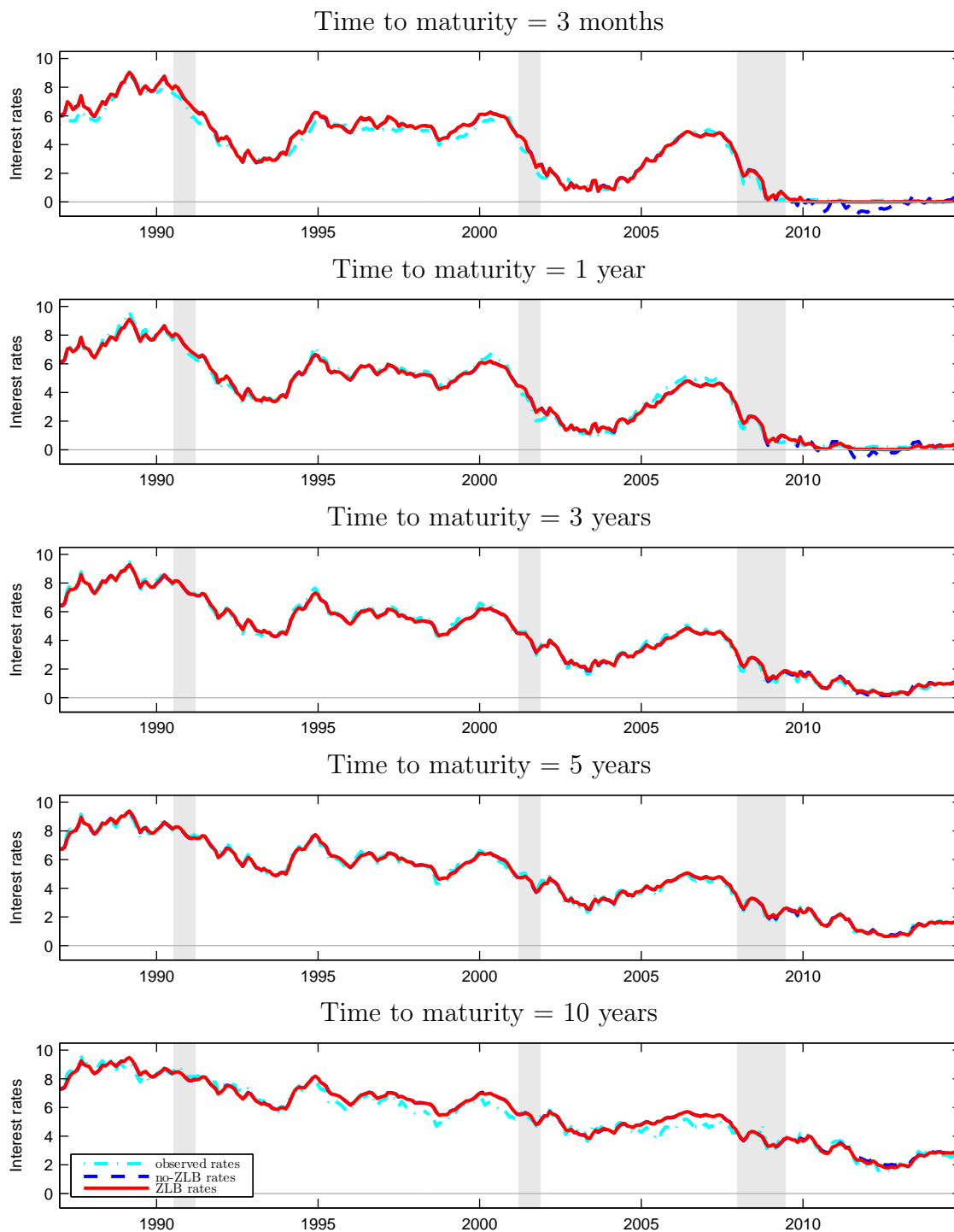
The table shows the unconditional moments of nominal bond yields in the data and implied by two types of economic models. The models labeled No ZLB and ZLB are described in Sections 3.2 and 3.3, respectively. In Panel A, Data refers to the complete data sample from 1987:M1 to 2014:M12, and the numbers shown for the models are computed from Monte Carlo simulations with length of 1,000,000 months. In Panel B, Data refers to the period from 2008-2014. For the two models ZLB months are identified by the 1-month shadow rate being negative. Model parameters are fixed at their median posterior estimates.

Figure 1: **Estimated Macroeconomic States**



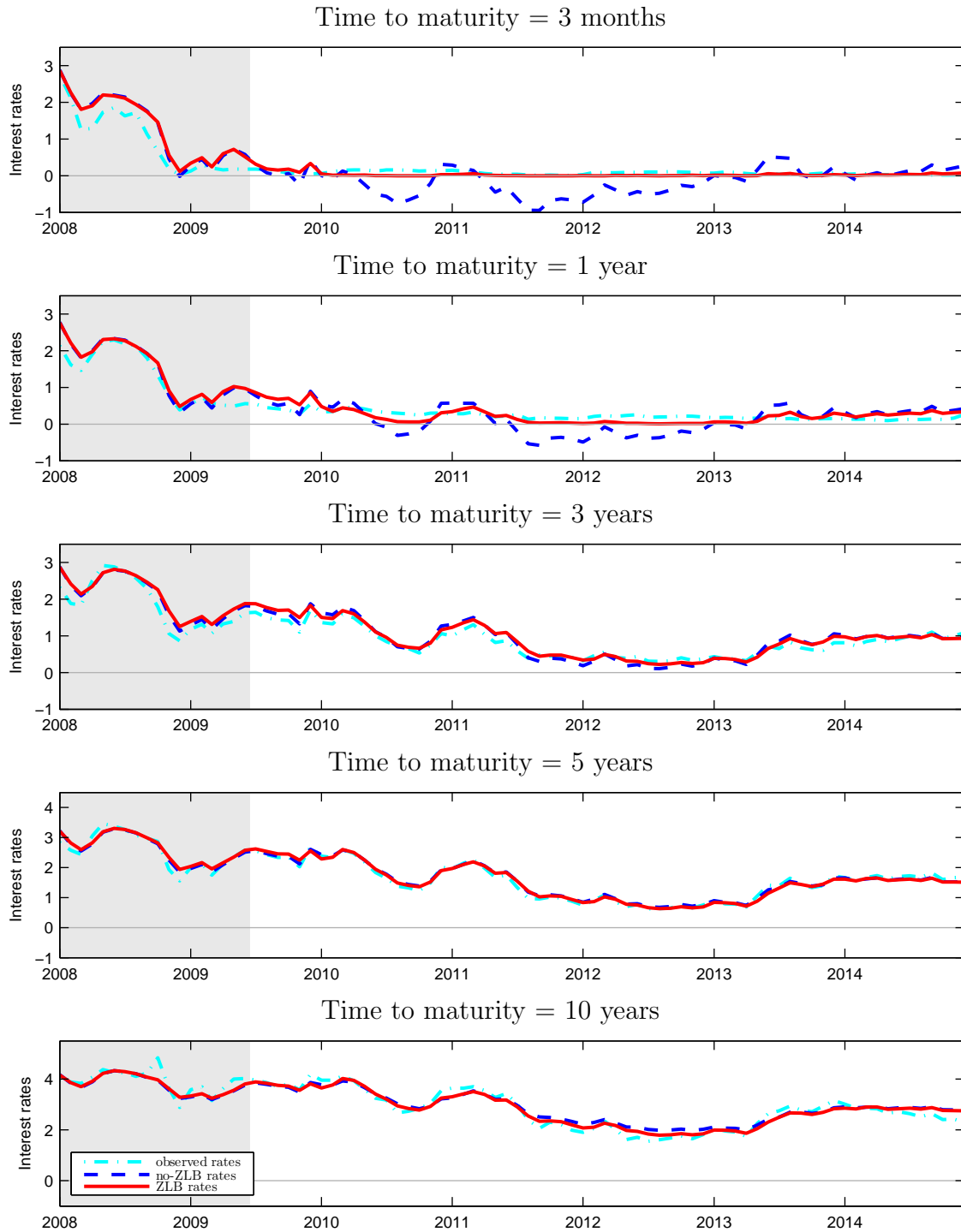
The figure shows the economic states estimated under the No ZLB and the ZLB model (solid and dashed lines, respectively). From top to bottom the graphs show expected consumption growth, expected inflation, and the volatility of expected inflation. The upper and middle graph also show the realized values of consumption growth and inflation (dashed-dotted lines). Data are from 1987:M1 to 2014:M12. Grey regions indicate NBER recessions. The variables are annualized, and in per cent.

Figure 2: Yields in the Data and in the Models



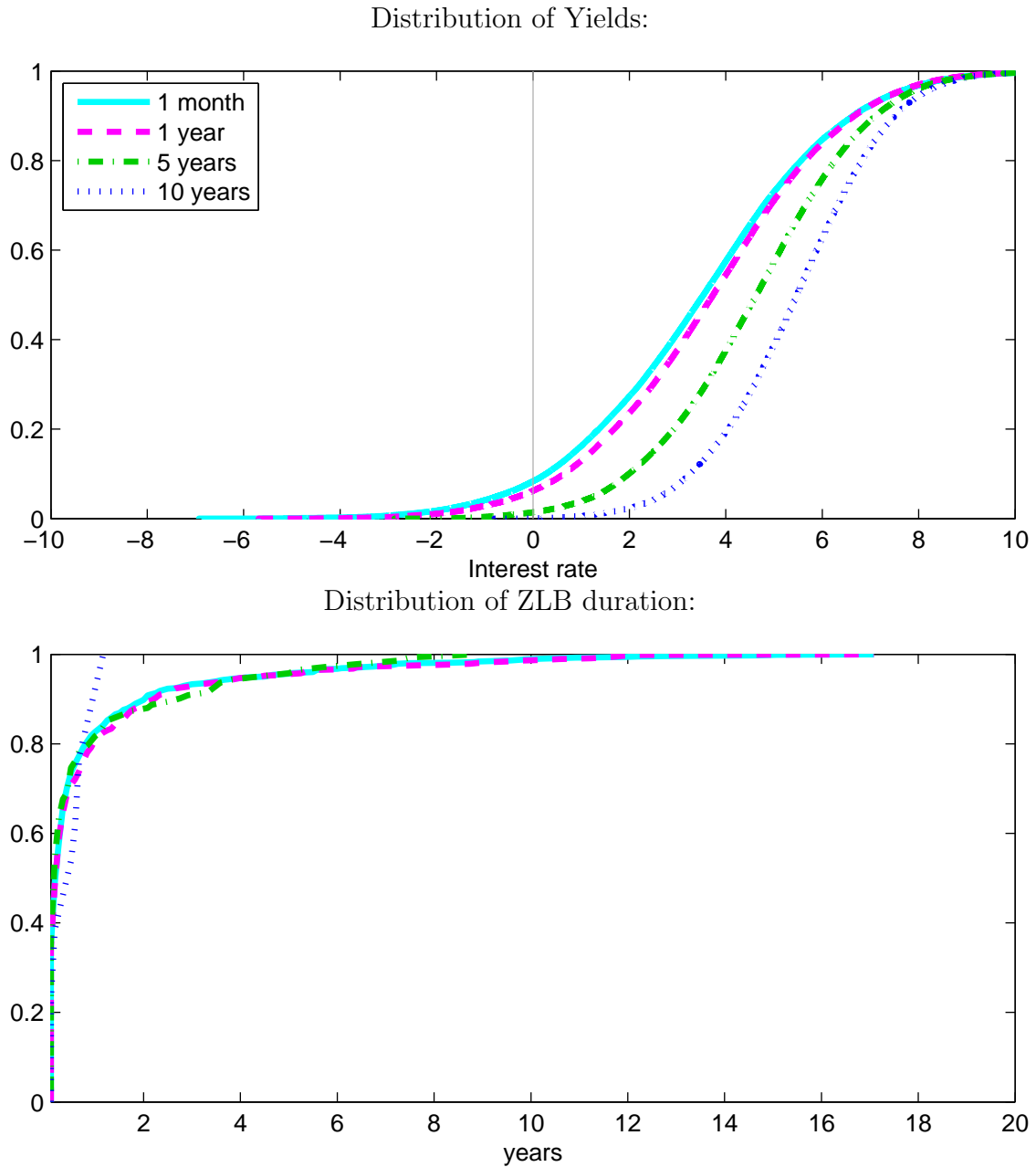
The figure shows the implied model yields under the no-ZLB model (dashed line), ZLB model (solid line), and yields in the data (dashed-dotted line). No-ZLB yields are evaluated from the no-ZLB model and using no-ZLB model state variables, while ZLB yields use ZLB model and ZLB model state variables. Data are from 1987:M1 to 2014:M12. Grey regions indicate NBER recessions.

Figure 3: Yields in the Data and in the Models: 2008-2014



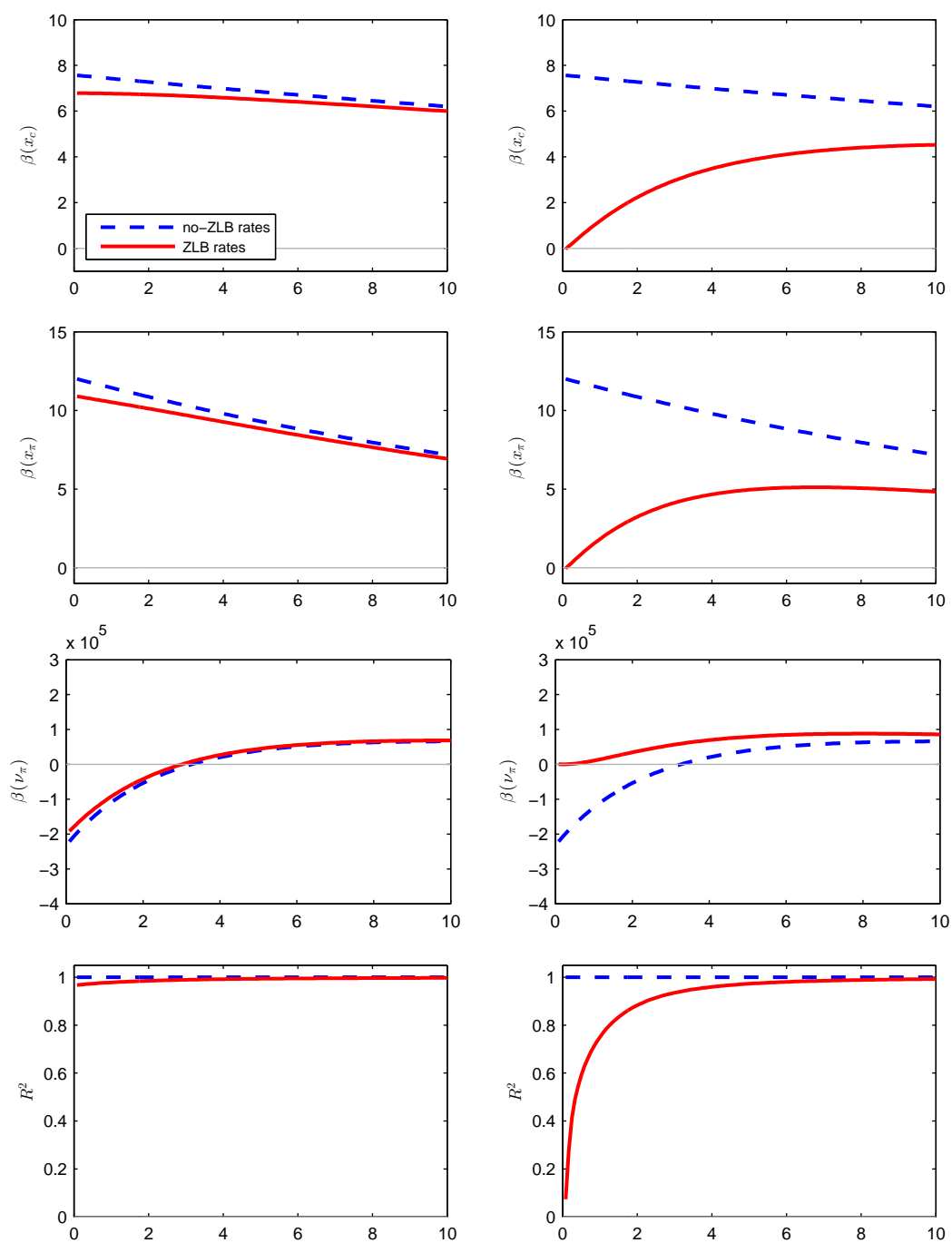
The figure shows the implied model yields under the no-ZLB model (dashed line), ZLB model (solid line), and yields in the data (dashed-dotted line), in the ZLB period. No-ZLB yields are evaluated from the no-ZLB model and using no-ZLB model state variables, while ZLB yields use ZLB model and ZLB model state variables. Data are from 2008:M1 to 2014:M12.

Figure 4: Model-Implied Distribution of Yields and ZLB Duration



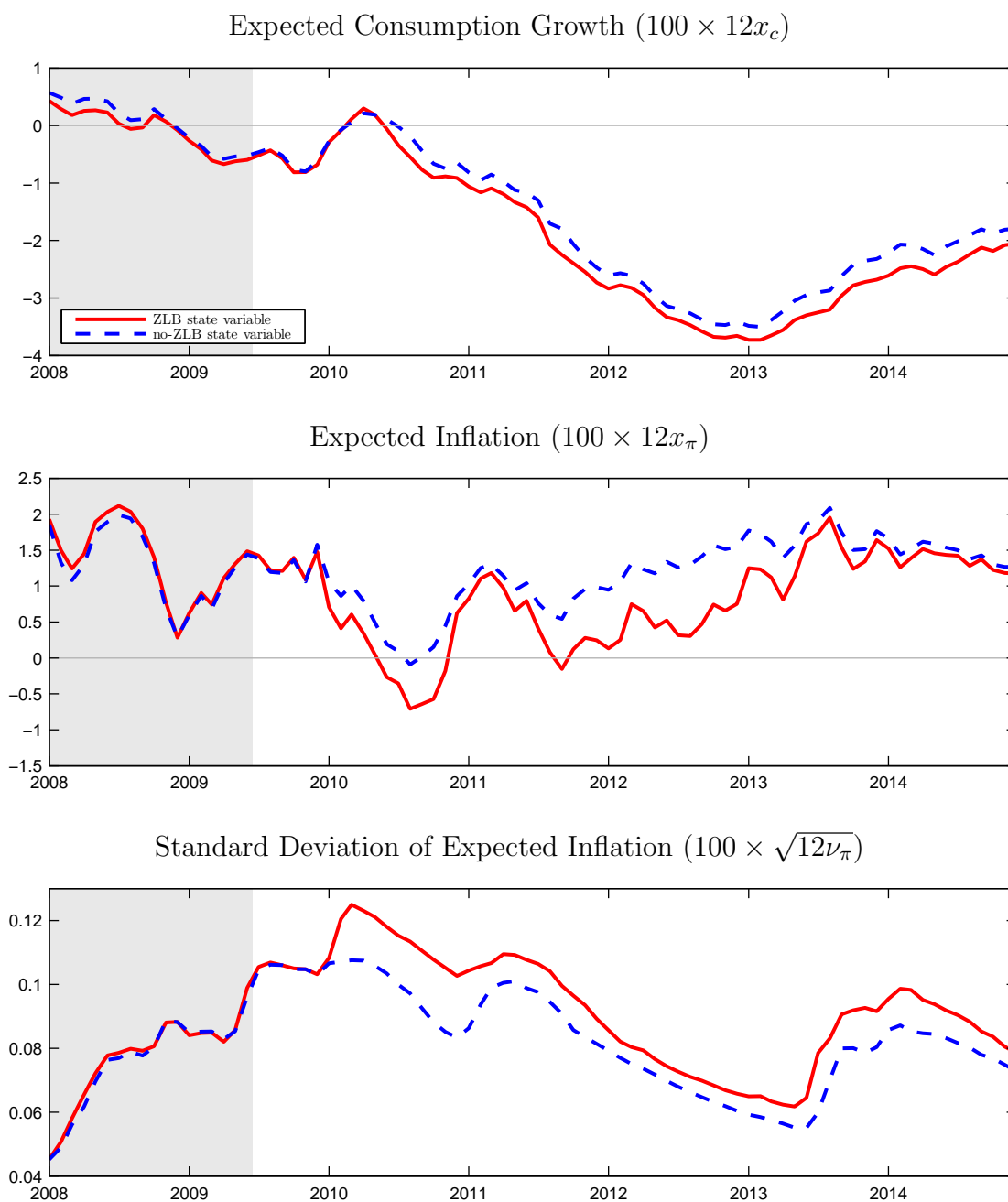
The figure shows the cumulative distribution functions for 1-month, 1-year, 5-year, and 10-year yields implied by the No ZLB model (upper graph) and the cumulative distribution function for the duration of zero lower bound periods for 1-month, 1-year, 5-year, and 10-year yields (lower graph).

Figure 5: Impact of the States on Yields



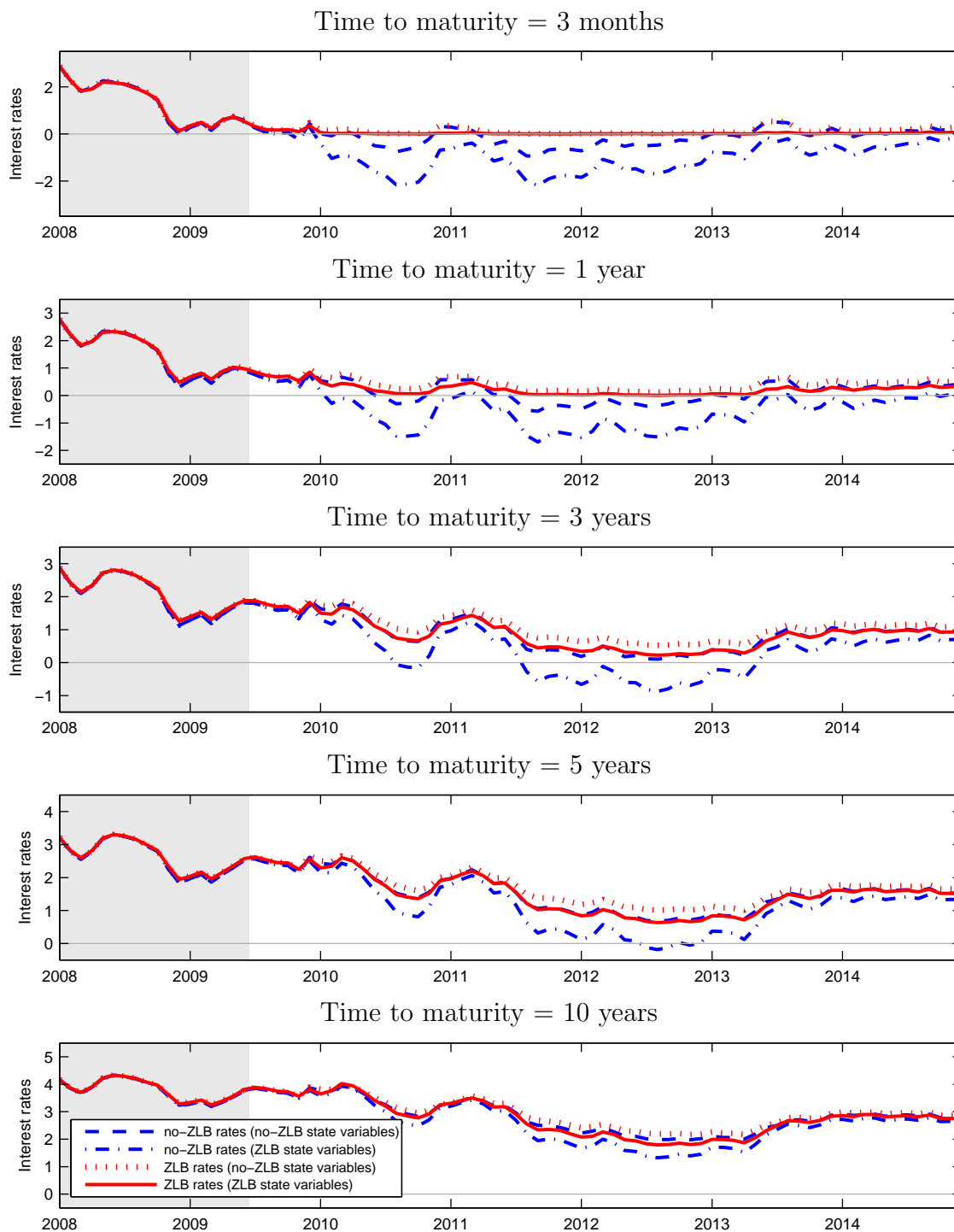
The figure shows linear loadings of No ZLB and ZLB yields on model state variables (from top to bottom: expected consumption growth, expected inflation, and volatility of expected inflation) as well the R^2 of the corresponding regression (bottom graph). The left column of graphs shows the results unconditionally, the right column shows the output for the zero lower bound periods.

Figure 6: **Estimated Economic States: 2008-2014**



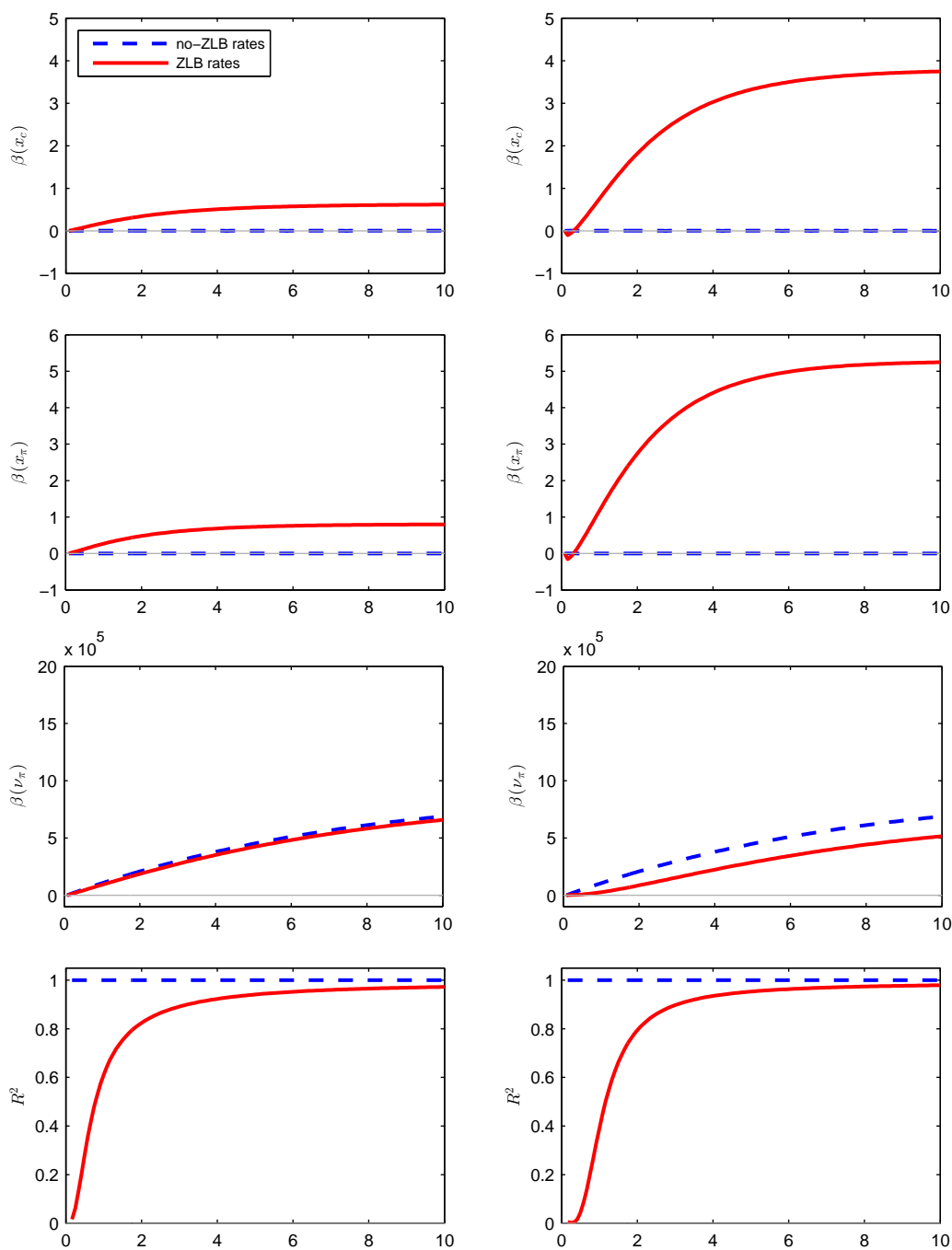
The figure shows the economic states estimated under the No ZLB and the ZLB model (solid and dashed lines, respectively). From top to bottom the graphs show expected consumption growth, expected inflation, and the volatility of expected inflation. Grey regions indicate NBER recessions. Data are from 2008:M1 to 2014:M12.

Figure 7: Model Yields: Implications of the ZLB and States



The figure shows the time-series of model-implied yields under the No ZLB model, evaluated at No ZLB and ZLB states (dashed and dashed-dotted lines); and under the ZLB model, evaluated at no-ZLB and ZLB states (dotted and solid lines). The yields are plotted over the 2008:M1 to 2014:M12 period.

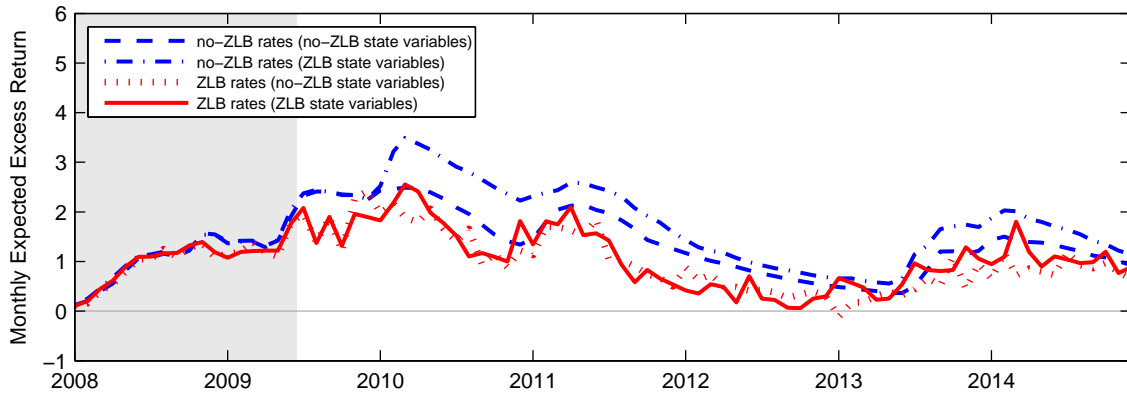
Figure 8: **Impact of the States on Expected Excess Returns**



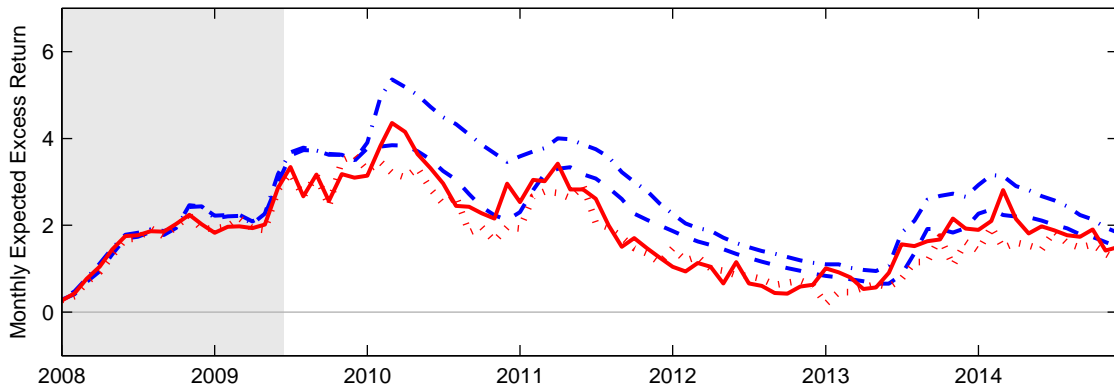
The figure shows linear loadings of No ZLB and ZLB monthly expected excess returns on model state variables (from top to bottom: expected consumption growth, expected inflation, and volatility of expected inflation) as well the R^2 of the corresponding regression (bottom graph). The left column of graphs shows the results unconditionally, the right column shows the output for the zero lower bound periods.

Figure 9: **Expected Excess Nominal Bond Returns**

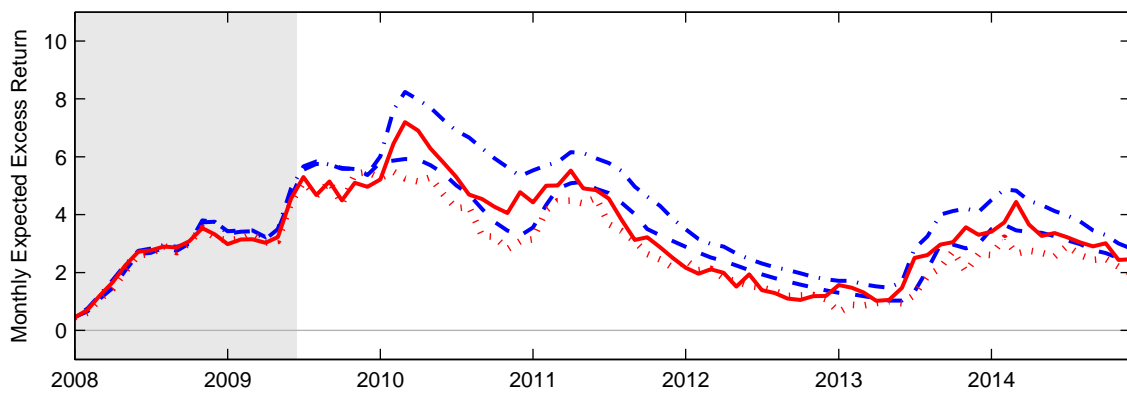
Time to maturity = 3 years



Time to maturity = 5 years

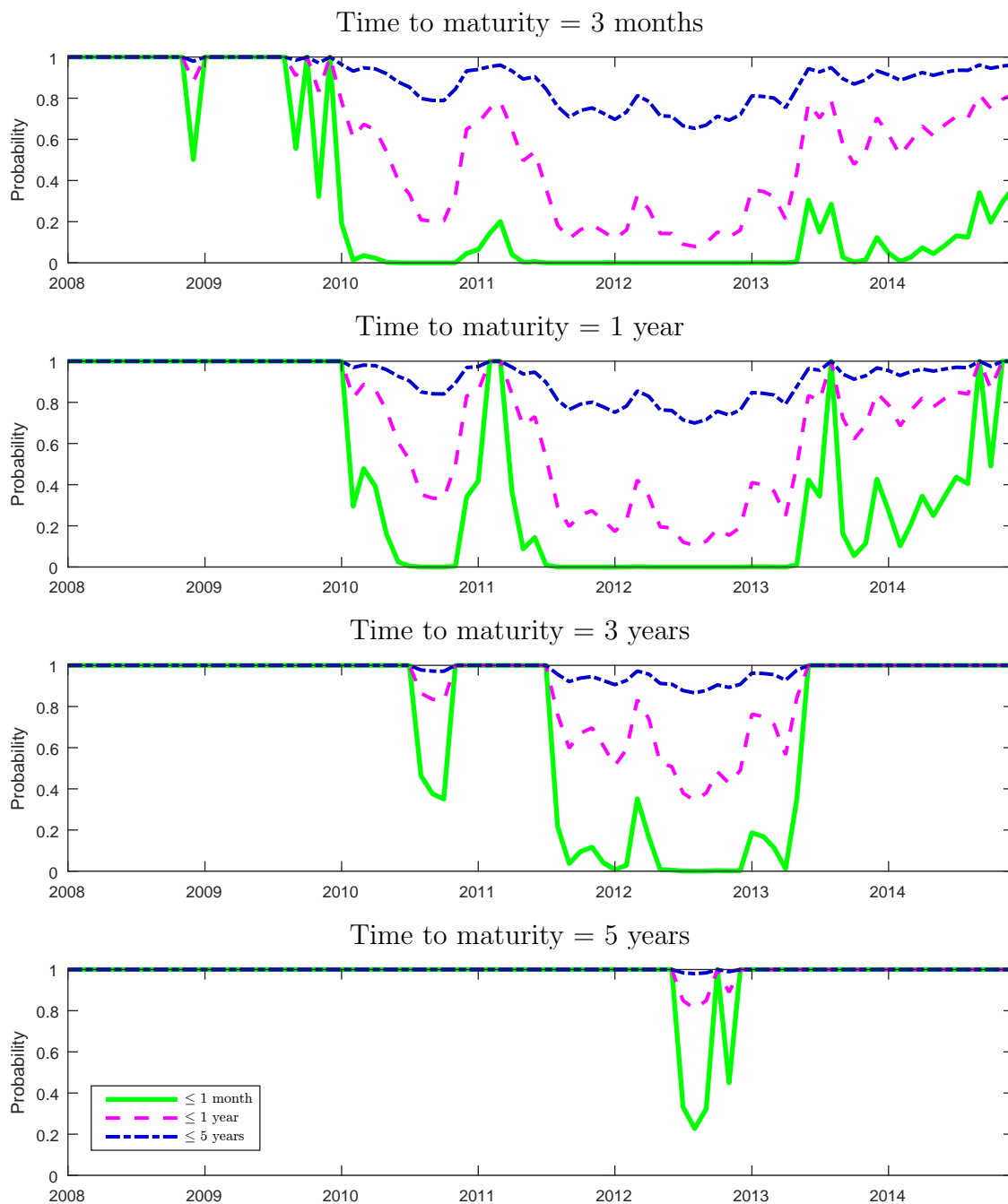


Time to maturity = 10 years



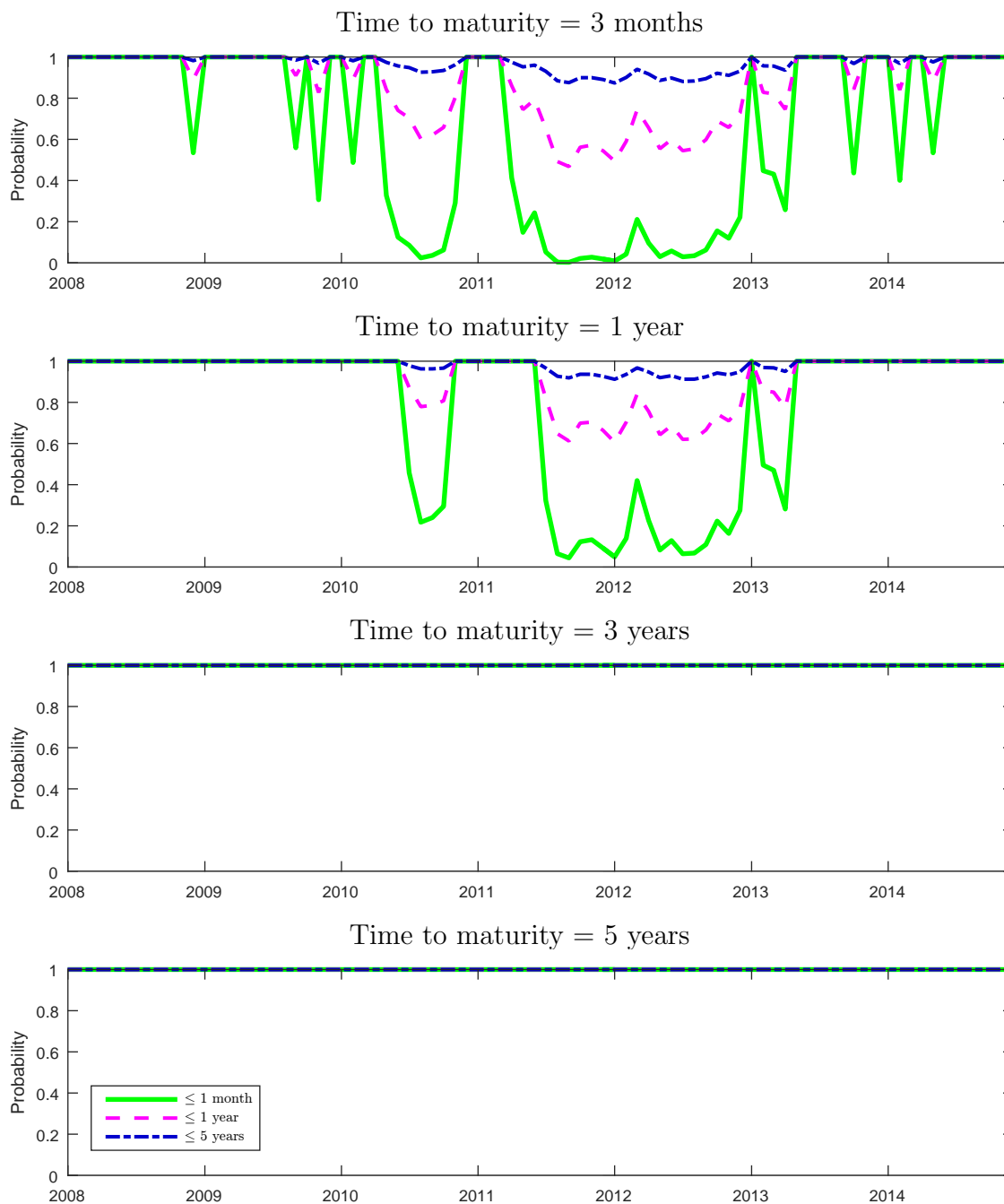
The figure shows the time-series of model-implied monthly expected excess nominal bond returns under the No ZLB model, evaluated at No ZLB and ZLB states (dashed and dashed-dotted lines); and under the ZLB model, evaluated at No ZLB and ZLB states (dotted and solid lines). The yields are plotted over the ZLB period.

Figure 10: **Lift-off Probabilities: ZLB State Variables**



The figure shows conditional lift-off probabilities for bond maturities of 3 months, 1, 3, and 5 years, and horizons of 1 months, 1, and 5 years. The probabilities are computed from the Monte-Carlo simulation with 10000 runs and using the ZLB state variables.

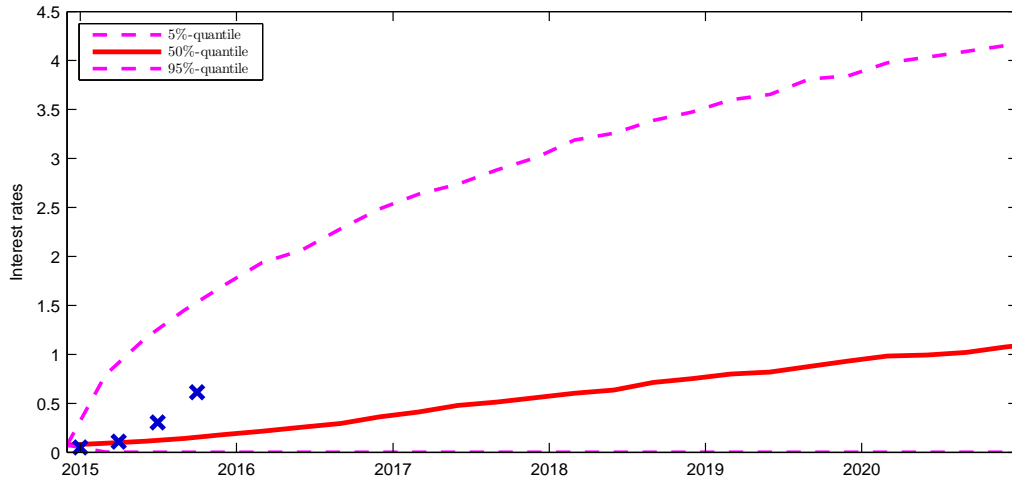
Figure 11: **Lift-off Probabilities: No ZLB State Variables**



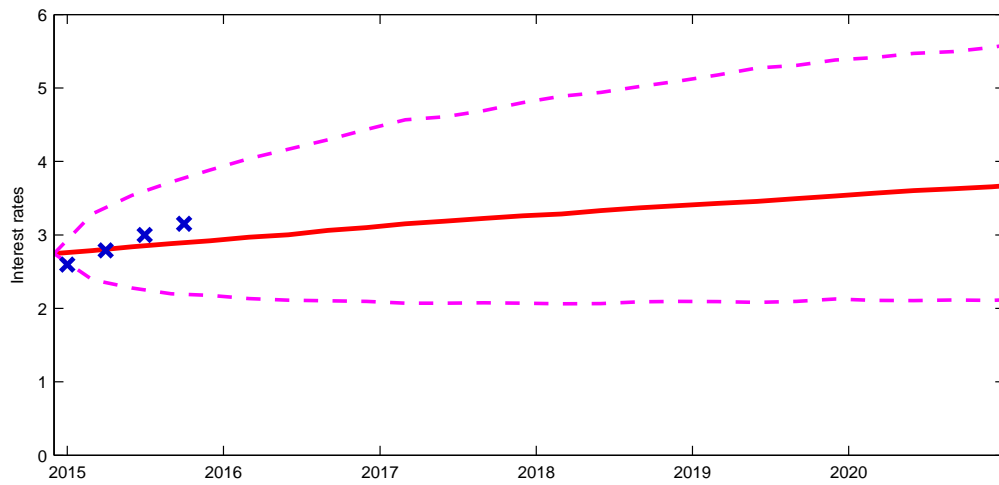
The figure shows conditional lift-off probabilities for bond maturities of 3 months, 1, 3, and 5 years, and horizons of 1 months, 1, and 5 years. The probabilities are computed from the Monte-Carlo simulation with 10000 runs and using the No ZLB state variables.

Figure 12: **Forecasts of Nominal ZLB-Yields**

Time to maturity = 3 months



Time to maturity = 10 years



The figure shows the forecasts of nominal yields at the end of the sample (2014:M12). It shows the median, the 5%-quantile, and the 95%-quantile based on a Monte-Carlo simulation with 1000 runs. The data (crosses) show yield forecasts from the Survey of Professional Forecasters.